

Intelligent Technique of Constructing Exact Statistical Tolerance Limits to Predict Future Outcomes under Parametric Uncertainty for Prognostics and Health Management of Complex Systems

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Abstract—The problem of constructing one-sided exact statistical tolerance limits on the k th order statistic in a future sample of m observations from a distribution of log-location-scale family on the basis of an observed sample from the same distribution is considered. The new technique proposed here emphasizes pivotal quantities relevant for obtaining tolerance factors and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. The exact tolerance limits on order statistics associated with sampling from underlying distributions can be found easily and quickly making tables, simulation, Monte Carlo estimated percentiles, special computer programs, and approximation unnecessary. Finally, numerical examples are given, where the tolerance limits obtained by using the known methods are compared with the results obtained through the proposed novel technique, which is illustrated in terms of the extreme-value and two-parameter Weibull distributions. The aim of this technique is to develop and publish original scientific contributions and industrial applications dealing with the topics covered by Prognostics and Health Management (PHM) of complex systems. PHM is a set of means, approaches, methods and tools that allows monitoring and tracking the health state of a system in order to detect, diagnose and predict its failures. This information is then exploited to take appropriate decisions to increase the system's availability, reliability and security while reducing its maintenance costs. The proposed technique allows one to construct developments and results in the areas of condition monitoring, fault detection, fault diagnostics, fault prognostics and decision support.

Keywords—future outcome, extreme-value distribution, two-parameter Weibull distribution, parametric uncertainty, Type II censored data, statistical tolerance limits, prognostics and health management of complex systems.

I. Introduction

The logical purpose for a statistical tolerance limit (where the coverage value γ is the percentage of the future process outcomes to be captured by the prediction, and the confidence level $(1-\alpha)$ is the proportion of the time we hope to capture that percentage γ) is to predict future outcomes for some production process which is treated as process, say, with stochastic variation of a product lifetime. The applications of tolerance limits (intervals) are varied. They included clinical and industrial applications, including

quality control, applications to environmental monitoring, to the assessment of agreement between two methods or devices, and applications in industrial hygiene. For example, such tolerance limits are required, when planning life tests, engineers may need to predict the number of failures that will occur by the end of the test or to predict the amount of time that it will take for a specified number of units to fail. Tolerance limits of the type mentioned above are considered in this paper, which presents a new technique for constructing exact statistical (lower and upper) tolerance limits on outcomes (for example, on order statistics) in future samples. Attention is restricted to the extreme-value and two-parameter Weibull distributions under parametric uncertainty (when both parameters are unknown). The technique used here emphasizes pivotal quantities relevant for obtaining tolerance factors and is applicable whenever the statistical problem is invariant under a group of transformations that acts transitively on the parameter space. It does not require the construction of any tables and is applicable whether the experimental data are complete or Type II censored. The exact tolerance limits on order statistics associated with sampling from underlying distributions can be found easily and quickly making tables, simulation, Monte Carlo estimated percentiles, special computer programs, and approximation unnecessary. The proposed technique is based on a probability transformation and pivotal quantity averaging. It does not in need to make any assumption concerning the statistical functional form for the tolerance limit, is conceptually simple and easy to use. The scientific literature does not contain an analytical methodology for constructing exact γ -content tolerance limits with expected $(1-\alpha)$ -confidence on future order statistics coming from an extreme-value or Weibull distribution. One reason is that the theoretical concept and computational complexity of the tolerance limits is significantly more difficult than that of the standard confidence and prediction limits. However, in the literature there are several known methods for constructing $(1-\alpha)$ -prediction limits (in terms of this paper, tolerance limits with expected $(1-\alpha)$ -confidence) on future order statistics coming from the two-parameter Weibull distribution. Therefore, finally, we give numerical examples, where the $(1-\alpha)$ -prediction limits obtained by using the known methods are compared with the results obtained through the proposed analytical methodology, which is illustrated in terms of the

extreme-value and two-parameter Weibull distributions. Analytical formulas for the tolerance limits are available in the scientific literature for only simple cases, for example, for the upper or lower tolerance limit for a univariate normal population. Thus it becomes necessary to use new methods in order to derive exact statistical tolerance limits for many populations. The proposed, in this paper, technique of intelligent constructing exact statistical γ -content tolerance limits with expected $(1-\alpha)$ -confidence, which are obtained here in terms of the two-parameter Weibull and extreme-value distributions, represents a novelty in the theory of statistical decisions.

II. Background

In this paper, two types of statistical tolerance limits are defined: i) γ -content tolerance limits with expected $(1-\alpha)$ -confidence on future outcomes, ii) tolerance limits with expected $(1-\alpha)$ -confidence on future outcomes. To be specific, let γ denote a proportion between 0 and 1. Then one-sided statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence is determined to capture a proportion γ or more of the population, with a given expected confidence level $1-\alpha$. For example, an upper statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence on future outcomes from a univariate population is such that with the given expected confidence level $1-\alpha$, a specified proportion γ or more of the population will fall below the limit. A lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence satisfies similar conditions. An upper statistical tolerance limit with expected $(1-\alpha)$ -confidence is determined so that the expected proportion of the population failing below the limit is $(1-\alpha)$. A lower statistical tolerance limit with expected $(1-\alpha)$ -confidence satisfies similar conditions. The statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence seems to be more useful than the statistical tolerance limit with expected $(1-\alpha)$ -confidence but is relatively difficult to construct.

The logical purpose for a tolerance limit must be the prediction of future outcomes for some (say, stochastic) process. Tolerance (prediction) limits enjoy a fairly rich history in the scientific literature and have a very important role in engineering and manufacturing applications. Patel [1] provides a review (which was fairly comprehensive at the time of publication) of tolerance intervals (limits) for many distributions as well as a discussion of their relation with confidence intervals (limits) for percentiles. Dunsmore [2] and Guenther, Patil, and Uppuluri [3] both discuss 2-parameter exponential tolerance intervals (limits) and the estimation procedure in greater detail. Engelhardt and Bain [4] discuss how to modify the formulas when dealing with Type II censored data. Guenther [5] and Hahn and Meeker [6] discuss how one-sided tolerance limits can be used to obtain approximate two-sided tolerance intervals by applying Bonferroni's inequality. In Nechval et al. [7-15], the exact statistical tolerance and prediction limits are discussed under parametric uncertainty of underlying models.

In contrast to other statistical limits commonly used for statistical inference, the γ -content tolerance limits with expected $(1-\alpha)$ -confidence (especially for the order statistics) are used relatively rarely. One reason is that the

theoretical concept and computational complexity of the γ -content tolerance limits with expected $(1-\alpha)$ -confidence is significantly more difficult than that of the standard confidence and prediction limits. Thus it becomes necessary to use the innovative approaches which will allow one to construct tolerance limits on future order statistics for many populations.

III. Focus of the Paper

A. Problem Statement

The problem can be stated more formally as follows. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable X from a sample of size n from a distribution with a probability density function $f_\theta(x)$ (distribution function $F_\theta(x)$, survival function $\bar{F}_\theta(x) = 1 - F_\theta(x)$) and S be any statistic (say, sufficient statistic or maximum likelihood estimator) obtained from the experimental random sample $X_1 \leq \dots \leq X_r$, and let a random variable Y (in a future random sample Y_1, \dots, Y_m) has the same distribution with the probability density function $f_\theta(y)$ (distribution function $F_\theta(y)$, survival function $\bar{F}_\theta(y) = 1 - F_\theta(y)$), where a parameter θ (in general, vector) is common to both distributions and it is assumed that some or all numerical values of components of the parametric vector θ are unspecified.

On the basis of the experimental random sample $X_1 \leq \dots \leq X_r$, we wish to make a prediction about a future outcome of Y_k (k th order statistic, $1 \leq k \leq m$, in a future random sample of m ordered observations $Y_1 \leq \dots \leq Y_m$), usually in the form of one-sided statistical tolerance limits on future outcomes of Y_k (lower γ -content tolerance limit L_k with expected $(1-\alpha)$ -confidence and upper γ -content tolerance limit U_k with expected $(1-\alpha)$ -confidence). That is, if L_k and U_k are functions of S , then $L_k \equiv L_k(S)$ is a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence on future outcomes of the k th order statistic Y_k if

$$E_\theta \left\{ \Pr \left(\int_{L_k(S)}^{\infty} g_\theta(y_k) dy_k \geq \gamma \right) \right\} = E_\theta \left\{ \Pr \left(\bar{G}_\theta(L_k(S)) \geq \gamma \right) \right\} = 1 - \alpha, \quad (1)$$

and $U_k \equiv U_k(S)$ is an upper statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence on future outcomes of the k th order statistic Y_k if

$$E_\theta \left\{ \Pr \left(\int_0^{U_k(S)} g_\theta(y_k) dy_k \geq \gamma \right) \right\} = E_\theta \left\{ \Pr \left(G_\theta(U_k(S)) \geq \gamma \right) \right\} = 1 - \alpha, \quad (2)$$

where

$$g_\theta(y_k) = \frac{1}{B(k, m-k+1)} [F_\theta(y_k)]^{k-1} [1 - F_\theta(y_k)]^{m-k} f_\theta(y_k) \quad (3)$$

is the probability density function of the k th order statistic Y_k ,

$$\begin{aligned}
 G_{\theta}(y_k) &= \Pr(Y_k \leq y_k) = \sum_{i=k}^m \binom{m}{i} [F_{\theta}(y_k)]^i [1 - F_{\theta}(y_k)]^{m-i} \\
 &= \sum_{i=k}^m \binom{m}{i} [1 - \bar{F}_{\theta}(y_k)]^i [\bar{F}_{\theta}(y_k)]^{m-i} \\
 &= \sum_{i=k}^m \binom{m}{i} \sum_{j=0}^i \binom{i}{j} (-1)^j [\bar{F}_{\theta}(y_k)]^{m-i+j} \\
 &= \int_0^{\bar{F}_{\theta}(y_k)} \frac{1}{B(k, m-k+1)} \tau^{k-1} (1-\tau)^{m-k} d\tau \quad (4)
 \end{aligned}$$

is the probability distribution function of the k th order statistic Y_k . It can be shown that

$$\frac{dG_{\theta}(y_k)}{dy_k} = g_{\theta}(y_k). \quad (5)$$

Further, $L_k^{\circ} \equiv L_k^{\circ}(S)$ is a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence on future outcomes of the k th order statistic Y_k from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ if it satisfies

$$\begin{aligned}
 E_{\theta} \left\{ \Pr(Y_k > L_k^{\circ}(S)) \right\} &= E_{\theta} \left\{ \int_{L_k^{\circ}(S)}^{\infty} g_{\theta}(y_k) dy_k \right\} \\
 &= E_{\theta} \left\{ \bar{G}_{\theta}(L_k^{\circ}(S)) \right\} = 1 - \alpha. \quad (6)
 \end{aligned}$$

$U_k^{\circ} \equiv U_k^{\circ}(S)$ is an upper statistical tolerance limit with expected $(1-\alpha)$ -confidence on future outcomes of the k th order statistic Y_k from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ if it satisfies

$$\begin{aligned}
 E_{\theta} \left\{ \Pr(Y_k \leq U_k^{\circ}(S)) \right\} &= E_{\theta} \left\{ \int_0^{U_k^{\circ}(S)} g_{\theta}(y_k) dy_k \right\} \\
 &= E_{\theta} \left\{ G_{\theta}(U_k^{\circ}(S)) \right\} = 1 - \alpha. \quad (7)
 \end{aligned}$$

In this paper, a new technique for intelligent constructing the statistical γ -content tolerance limits with expected $(1-\alpha)$ -confidence as well as the statistical tolerance limits with expected $(1-\alpha)$ -confidence on order statistics in future samples is proposed. For illustration, the extreme-value and Weibull distributions are considered.

B. Extreme-Value Distribution

This distribution is used in many research fields including, among others, life testing and water resource management. This is the so-called first asymptotic distribution of extreme values, hereafter referred to simply as the extreme-value distribution. The distribution is extensively used in a number of areas as a lifetime distribution and sometimes referred to as the Gumbel distribution, after E. J. Gumbel, who had pioneered its use (Gumbel [16]).

Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable X from a sample of size n from an extreme-value distribution with the pdf (probability density function),

$$\begin{aligned}
 f_{\theta}(x) &= \frac{1}{\theta_2} \exp\left(\frac{x-\theta_1}{\theta_2}\right) \exp\left[-\exp\left(\frac{x-\theta_1}{\theta_2}\right)\right], \\
 &\quad -\infty < x < \infty, \quad (8)
 \end{aligned}$$

and cdf (cumulative distribution function),

$$F_{\theta}(x) = 1 - \exp\left[-\exp\left(\frac{x-\theta_1}{\theta_2}\right)\right], \quad -\infty < x < \infty, \quad (9)$$

indexed by location and scale parameters θ_1 and θ_2 , where $\theta = (\theta_1, \theta_2)$. It is assumed that the parameters θ_1 ($-\infty < \theta_1 < \infty$) and $\theta_2 > 0$ are unknown.

In Type II censoring, which is of primary interest here, the number of survivors are fixed and X_r is a random variable. The MLE's $\hat{\theta}_1$ and $\hat{\theta}_2$ of the parameters θ_1 and θ_2 , respectively, are solutions of

$$\hat{\theta}_1 = \ln \left[r^{-1} \left(\sum_{i=1}^r \exp\left(\frac{x_i}{\hat{\theta}_2}\right) + (n-r) \exp\left(\frac{x_r}{\hat{\theta}_2}\right) \right) \right]^{\hat{\theta}_2}, \quad (10)$$

$$\begin{aligned}
 \hat{\theta}_2 &= \left[\sum_{i=1}^r x_i \exp\left(\frac{x_i}{\hat{\theta}_2}\right) + (n-r) x_r \exp\left(\frac{x_r}{\hat{\theta}_2}\right) \right] \\
 &\times \left[\sum_{i=1}^r \exp\left(\frac{x_i}{\hat{\theta}_2}\right) + (n-r) \exp\left(\frac{x_r}{\hat{\theta}_2}\right) \right]^{-1} - \frac{1}{r} \sum_{i=1}^r x_i. \quad (11)
 \end{aligned}$$

In terms of the extreme-value distribution variates, we have that

$$W_1 = \frac{\hat{\theta}_1 - \theta_1}{\theta_2}, \quad W_2 = \frac{\hat{\theta}_2}{\theta_2}, \quad W_3 = \frac{\hat{\theta}_1 - \theta_1}{\hat{\theta}_2} \quad (12)$$

are pivotal quantities. The probability density functions of the pivotal quantities do not depend on the parameters. It can be shown that the joint pdf of the pivotal quantities

$$W_1 = \frac{\hat{\theta}_1 - \theta_1}{\theta_2}, \quad W_2 = \frac{\hat{\theta}_2}{\theta_2}, \quad (13)$$

conditional on fixed

$$\mathbf{S}^{(r)} = (S_1, \dots, S_r), \quad (14)$$

where

$$S_i = \frac{X_i - \hat{\theta}_1}{\hat{\theta}_2}, \quad i = 1, \dots, r, \quad (15)$$

are ancillary statistics, any $r-2$ of which form a functionally independent set, $\hat{\theta}_1$ and $\hat{\theta}_2$ are the maximum likelihood estimates for θ_1 and θ_2 , respectively, based on the first r ordered observations ($X_1 \leq \dots \leq X_r$) from a sample of size n from the extreme-value distribution (8), which can be found from solution of (10) and (11), is given by

$$f_n(w_1, w_2 | \mathbf{s}^{(r)}) = \left(\frac{(e^{w_1})^r}{\Gamma(r)} \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right] \right)^r$$

$$\begin{aligned} & \times \exp \left(-e^{w_1} \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right] \right) \\ & \times \left(\frac{1}{v(s^{(r)})} w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2) \right) \\ & \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^{-r} \\ & = f_n(w_1 | s^{(r)}, w_2) f_n(w_2 | s^{(r)}), \\ & w_1 \in (-\infty, \infty), \quad w_2 \in (0, \infty), \end{aligned} \quad (16)$$

where

$$\begin{aligned} f_n(w_1 | s^{(r)}, w_2) &= \frac{(e^{w_1})^r}{\Gamma(r)} \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^r \\ & \times \exp \left(-e^{w_1} \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right] \right), \\ & w_1 \in (-\infty, \infty). \\ f_n(w_2 | s^{(r)}) &= \frac{1}{v(s^{(r)})} w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2) \\ & \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^{-r}, \quad w_2 \in (0, \infty), \quad (18) \\ v(s^{(r)}) &= \int_0^\infty w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2) \\ & \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^{-r} dw_2 \quad (19) \end{aligned}$$

is the normalizing constant. If a pivotal quantity is given by

$$W = e^{w_1} \left[\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2) \right], \quad (20)$$

it follows from (17) that

$$W \sim g_r(w) = \frac{1}{\Gamma(r)} w^{r-1} \exp(-w), \quad w \in (0, \infty). \quad (21)$$

C. Two-Parameter Weibull Distribution

The two-parameter Weibull distribution is one of the most widely used life distributions in reliability analysis. This distribution is very flexible, and can, through an appropriate choice of parameters, model many types of failure rate behaviors. It has wide applications in diverse disciplines.

Let $\underline{X}_1 \leq \dots \leq \underline{X}_r$ be the first r ordered observations of a random variable \underline{X} from a sample of size n from a two-parameter Weibull distribution with the pdf,

$$f_{\underline{X}}(\underline{x}) = \frac{\delta}{\beta} \left(\frac{\underline{x}}{\beta} \right)^{\delta-1} \exp \left[-\left(\frac{\underline{x}}{\beta} \right)^{\delta} \right], \quad \underline{x} > 0, \quad \beta > 0, \quad \delta > 0, \quad (22)$$

and cdf,

$$F_{\underline{X}}(\underline{x}) = 1 - \exp \left[-\left(\frac{\underline{x}}{\beta} \right)^{\delta} \right], \quad \underline{x} > 0, \quad \beta > 0, \quad \delta > 0, \quad (23)$$

indexed by scale and shape parameters β and δ , where $\underline{\theta} = (\beta, \delta)$. It is assumed that the parameters β and δ are unknown. This distribution is directly related to the extreme-value distribution by the easily shown fact that if \underline{X} has a Weibull distribution (22), then $X = \ln \underline{X}$ has an extreme-value distribution with $\theta_1 = \ln \beta$ and $\theta_2 = \delta^{-1}$. In analyzing data it is often convenient to work with log times, the extreme-value distribution arises when lifetimes are taken to be Weibull distributed. The MLE's of the Weibull parameters β and δ are $\hat{\beta} = \exp \hat{\theta}_1$ and $\hat{\delta} = \hat{\theta}_2^{-1}$. If desired, the maximum likelihood equations (10) and (11) can be written in Weibull form and solved directly from the start. The equations are

$$\hat{\beta} = \left(r^{-1} \left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} + (n-r) \underline{x}_r^{\hat{\delta}} \right) \right)^{1/\hat{\delta}}, \quad (24)$$

$$\hat{\delta} = \left[\frac{\left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} \ln \underline{x}_i + (n-r) \underline{x}_r^{\hat{\delta}} \ln \underline{x}_r \right)}{\left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} + (n-r) \underline{x}_r^{\hat{\delta}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^r \ln \underline{x}_i} \right]^{-1}. \quad (25)$$

In terms of the Weibull variates, we have that

$$V_1 = \left(\frac{\hat{\beta}}{\beta} \right)^{\delta}, \quad V_2 = \frac{\delta}{\hat{\delta}}, \quad V_3 = \left(\frac{\hat{\beta}}{\beta} \right)^{\hat{\delta}} \quad (26)$$

are pivotal quantities. The probability density functions of the pivotal quantities do not depend on the parameters. It can be shown that the joint pdf of the pivotal quantities

$$V_1 = \left(\frac{\hat{\beta}}{\beta} \right)^{\delta}, \quad V_2 = \frac{\delta}{\hat{\delta}}, \quad (27)$$

conditional on fixed

$$\mathbf{Z}^{(r)} = (Z_1, \dots, Z_r), \quad (28)$$

where

$$Z_i = \left(\frac{X_i}{\beta} \right)^{\delta}, \quad i = 1, \dots, r, \quad (29)$$

are ancillary statistics, any $r-2$ of which form a functionally independent set, $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimates for β and δ , respectively, based on the first r ordered observations $X_1 \leq \dots \leq X_r$ from a sample of size n from the two-parameter Weibull distribution (22), which can be found from solution of (24) and (25), is given by

$$\begin{aligned} f_n(v_1, v_2 | \mathbf{z}^{(r)}) &= \left(\frac{v_1^{r-1}}{\Gamma(r)} \left[\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right]^r \right. \\ &\quad \times \exp \left(-v_1 \left[\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right] \right) \\ &\quad \times \left(\frac{1}{\mathcal{G}(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left[\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right]^{-r} \right) \\ &= f_n(v_1 | \mathbf{z}^{(r)}, v_2) f_n(v_2 | \mathbf{z}^{(r)}), \quad v_1 \in (0, \infty), \quad v_2 \in (0, \infty), \quad (30) \end{aligned}$$

where

$$\begin{aligned} f_n(v_1 | \mathbf{z}^{(r)}, v_2) &= \frac{1}{\Gamma(r)} \left[\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right]^r \\ &\quad \times v_1^{r-1} \exp \left(-v_1 \left[\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right] \right), \quad v_1 \in (0, \infty), \quad (31) \end{aligned}$$

$$f_n(v_2 | \mathbf{z}^{(r)}) = \frac{1}{\mathcal{G}(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right)^{-r},$$

$$v_2 \in (0, \infty), \quad (32)$$

$$\mathcal{G}(\mathbf{z}^{(r)}) = \int_0^\infty v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right)^{-r} dv_2 \quad (33)$$

is the normalizing constant. If a pivotal quantity is given by

$$V = V_1 \left[\sum_{i=1}^r Z_i^{V_2} + (n-r)Z_r^{V_2} \right], \quad (34)$$

it follows from (31) that

$$V \sim g_r(v) = \frac{1}{\Gamma(r)} v^{r-1} \exp(-v), \quad v \in (0, \infty). \quad (35)$$

IV. Constructing Statistical γ -Content Tolerance Limits with Expected $(1-\alpha)$ -Confidence

A. Constructing Lower Statistical γ -Content Tolerance Limit with Expected $(1-\alpha)$ -Confidence

Theorem 1. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable $X (= \ln \underline{X})$ from a sample of size n from an extreme-value distribution defined by the probability density function (8). Then a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $L_k \equiv L_k(S)$, on future outcomes of the k th order statistic $Y_k (= \ln \underline{Y}_k)$ from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ also from the distribution (8), which satisfies (1) is given by

$$L_k = \hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k} = \ln \underline{L}_k, \quad (36)$$

where η_{L_k} is a tolerance factor determined by

$$\begin{aligned} \eta_{L_k} &= \arg \left(\frac{\int_0^\infty \int_0^\infty \frac{\ln(1-q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right)}{[\eta_{L_k}]^{w_2}} \frac{1}{\Gamma(r)} w^{r-1} \exp(-w)}{v(\mathbf{s}^{(r)}) w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2)} \right. \\ &\quad \times \left. \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^{-r} dw dw_2 = 1 - \alpha \right); \quad (37) \end{aligned}$$

the maximum likelihood estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of the parameters θ_1 and θ_2 are determined from (10) and (11), respectively; the ancillary statistics S_i , $i=1, \dots, r$, are given by (15); $q_{1-\gamma}$ is a quantile of the beta distribution satisfying

$$\int_0^{q_{1-\gamma}} \frac{1}{B(k, m-k+1)} \tau^{k-1} (1-\tau)^{m-k} d\tau = 1 - \gamma. \quad (38)$$

Proof. It follows from (3) and (4) that

$$\begin{aligned} \Pr \left(\int_{L_k}^\infty g_\theta(y_k) dy_k \geq \gamma \right) &= \Pr \left(\bar{G}_\theta(L_k) \geq \gamma \right) = \Pr \left(1 - G_\theta(L_k) \geq \gamma \right) \\ &= \Pr \left(1 - \int_0^{F_\theta(L_k)} \frac{1}{B(k, m-k+1)} \tau^{k-1} (1-\tau)^{m-k} d\tau \geq \gamma \right) \\ &= \Pr \left(\int_0^{F_\theta(L_k)} \frac{1}{B(k, m-k+1)} \tau^{k-1} (1-\tau)^{m-k} d\tau \leq 1 - \gamma \right) \end{aligned}$$

$$= \Pr(F_{\theta}(L_k) \leq q_{1-\gamma}) = \Pr\left(1 - \exp\left[-\exp\left(\frac{L_k - \theta_1}{\theta_2}\right)\right] \leq q_{1-\gamma}\right) \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2)\right]^{-r} dw dw_2 = 1 - \alpha. \quad (40)$$

$$= \Pr\left(\exp\left[-\exp\left(\frac{L_k - \theta_1}{\theta_2}\right)\right] \geq 1 - q_{1-\gamma}\right)$$

$$= \Pr\left(e^{W_1} \leq \frac{\ln(1 - q_{1-\gamma})^{-1}}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}\right)$$

$$= \Pr\left(\frac{e^{W_1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2)\right)}{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2)\right)} \leq \frac{1}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}\right)$$

$$= \Pr\left(W \leq \frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}\right). \quad (39)$$

Using pivotal quantity averaging, it follows from (1) and (39) that

$$\begin{aligned} & E_{\theta} \left\{ \Pr \left(\int_{L_k(S)}^{\infty} g_{\theta}(y_k) dy_k \geq \gamma \right) \right\} \\ &= E \left\{ \Pr \left(W \leq \frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}} \right) \right\} \\ &= E \left\{ \int_0^{\frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}} g_r(w) dw \right\} \\ &= \int_0^{\infty} \int_0^{\frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}} g_r(w) f_n(v_2 | \mathbf{z}^{(r)}) dw dv_2 \\ &= \int_0^{\infty} \int_0^{\frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}} \frac{1}{\Gamma(r)} w^{r-1} \exp(-w) \\ &\quad \times \frac{1}{v(\mathbf{s}^{(r)})} w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2) \end{aligned}$$

It follows from (40) that

$$\begin{aligned} & L_k \\ &= \arg \left(\frac{\int_0^{\infty} \int_0^{\frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2)\right)}{\left[\exp((L_k - \hat{\theta}_1)/\hat{\theta}_2)\right]^{W_2}}} \frac{1}{\Gamma(r)} w^{r-1} \exp(-w)}{\times \frac{1}{v(\mathbf{s}^{(r)})} w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2)} \right. \\ &\quad \left. \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2)\right]^{-r} dw dv_2 = 1 - \alpha \right) \end{aligned} \quad (41)$$

Assuming that

$$\exp\left(\frac{L_k - \hat{\theta}_1}{\hat{\theta}_2}\right) = \eta_{L_k}, \quad (42)$$

we have (36). This completes the proof.

Theorem 2. Let $\underline{X}_1 \leq \dots \leq \underline{X}_r$ be the first r ordered observations of a random variable $\underline{X} (= \exp X)$ from a sample of size n from a two-parameter Weibull distribution defined by the probability density function (22). Then a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k(\underline{S})$, on future outcomes of the k th order statistic $\underline{Y}_k (= \exp Y_k)$ from a set of m future ordered observations $\underline{Y}_1 \leq \dots \leq \underline{Y}_m$ also from the distribution (22), which satisfies (1) is given by

$$\underline{L}_k = \eta_{L_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k, \quad (43)$$

where η_{L_k} is a tolerance factor determined by

$$\eta_{L_k} = \arg \left(\frac{\int_0^{\infty} \int_0^{\frac{\ln(1 - q_{1-\gamma})^{-1} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2}\right)}{\left[\eta_{L_k}\right]^{v_2}}} \frac{1}{\Gamma(r)} v^{r-1} \exp(-v)}{\times \frac{1}{g(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2}\right)^{-r}} dv dv_2 = 1 - \alpha \right); \quad (44)$$

the maximum likelihood estimates $\hat{\beta}$ and $\hat{\delta}$ of the parameters β and δ are determined from (24) and (25), respectively; the ancillary statistics Z_i , $i=1, \dots, r$, are given by (29); $q_{1-\gamma}$ is a quantile of the beta distribution satisfying (38).

Proof. The proof is similar to that of Theorem 1 and so it is omitted here.

Inference 1 (for the tolerance factors η_{L_k} and η_{U_k}). It follows from (42) and (43) that

$$\eta_{L_k} = \left(\frac{L_k}{\hat{\beta}} \right)^{\hat{\delta}} = \left(\frac{\exp L_k}{\exp \hat{\theta}_1} \right)^{1/\hat{\theta}_2} = \exp \left(\frac{L_k - \hat{\theta}_1}{\hat{\theta}_2} \right) = \eta_{L_k}. \quad (45)$$

B. Constructing Upper Statistical γ -Content Tolerance Limit with Expected $(1-\alpha)$ -Confidence

Theorem 3. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable $X (= \ln \underline{X})$ from a sample of size n from an extreme-value distribution defined by the probability density function (8). Then an upper statistical γ -content tolerance limit with $(1-\alpha)$ -confidence, $U_k \equiv U_k(S)$, on future outcomes of the k th order statistic $Y_k (= \ln Y_k)$ from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ also from the distribution (8), which satisfies (2) is given by

$$U_k = \hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{U_k} = \ln \underline{U}_k, \quad (46)$$

where η_{U_k} is a tolerance factor determined by

$$\eta_{U_k} = \arg \left[\int_0^\infty \frac{\ln(1-q_\gamma)^{-1} \left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right)}{[\eta_{U_k}]^{w_2}} \frac{1}{\Gamma(r)} w^{r-1} \exp(-w) \right. \\ \left. \times \frac{1}{v(s^{(r)})} w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2) \right. \\ \left. \times \left[\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right]^{-r} dw dw_2 = \alpha \right]; \quad (47)$$

the maximum likelihood estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ of the parameters θ_1 and θ_2 are determined from (10) and (11), respectively; the ancillary statistics S_i , $i=1, \dots, r$, are given by (15); q_γ is a quantile of the beta distribution satisfying (38).

Proof. The upper statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $U_k \equiv U_k(S)$, is obtained from a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $L_k \equiv L_k(S)$, by replacing γ by $1-\gamma$, and $1-\alpha$ by α . This completes the proof.

Theorem 4. Let $\underline{X}_1 \leq \dots \leq \underline{X}_r$ be the first r ordered observations of a random variable $\underline{X} (= \exp X)$ from a sample of size n from a two-parameter Weibull distribution defined by the probability density function (22). Then an upper statistical γ -content tolerance limit with $(1-\alpha)$ -confidence, $\underline{U}_k \equiv \underline{U}_k(S)$, on future outcomes of the k th

order statistic $\underline{Y}_k (= \exp Y_k)$ from a set of m future ordered observations $\underline{Y}_1 \leq \dots \leq \underline{Y}_m$ also from the distribution (22), which satisfies (2) is given by

$$\underline{U}_k = \eta_{\underline{U}_k}^{1/\hat{\delta}} \hat{\beta} = \exp U_k, \quad (48)$$

where $\eta_{\underline{U}_k}$ is a tolerance factor determined by

$$\eta_{\underline{U}_k} = \arg \left[\int_0^\infty \frac{\ln(1-q_\gamma)^{-1} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} \right)}{[\eta_{\underline{U}_k}]^{v_2}} \frac{1}{\Gamma(r)} v^{r-1} \exp(-v) \right. \\ \left. \times \frac{1}{g(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r} dv dv_2 = \alpha \right]. \quad (49)$$

Proof. The upper statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{U}_k \equiv \underline{U}_k(S)$, is obtained from a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k(S)$, by replacing γ by $1-\gamma$, and $1-\alpha$ by α . This completes the proof.

Inference 2 (for the tolerance factors $\eta_{\underline{U}_k}$ and η_{U_k}). It follows from (46) and (48) that

$$\eta_{\underline{U}_k} = \left(\frac{\underline{U}_k}{\hat{\beta}} \right)^{\hat{\delta}} = \left(\frac{\exp U_k}{\exp \hat{\theta}_1} \right)^{1/\hat{\theta}_2} = \exp \left(\frac{U_k - \hat{\theta}_1}{\hat{\theta}_2} \right) = \eta_{U_k}. \quad (50)$$

v. Constructing Statistical Tolerance Limits with Expected $(1-\alpha)$ -Confidence

A. Constructing Lower Statistical Tolerance Limit with Expected $(1-\alpha)$ -Confidence

Theorem 5. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable $X (= \ln \underline{X})$ from a sample of size n from an extreme-value distribution defined by the probability density function (8). Then a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $L_k \equiv L_k(S)$, on future outcomes of the k th order statistic $Y_k (= \ln Y_k)$ from a set of m future ordered observations $Y_1 \leq \dots \leq Y_m$ also from the distribution (8), which satisfies (6) is given by

$$L_k = \hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k} = \ln \underline{L}_k, \quad (51)$$

where η_{L_k} is a tolerance factor determined by

$$\eta_{l_k} = \arg \left\{ \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{\nu(\mathbf{s}^{(r)})} \right. \\ \left. \times \int_0^\infty \frac{w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2)}{\left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right)^r} dw_2 \right\} = 1 - \alpha. \quad (52)$$

Proof. It follows from (3) and (4) that

$$\begin{aligned}
\Pr(Y_k > L_k^\circ) &= \int_{L_k^\circ}^{\infty} g_\theta(y_k) dy_k = \bar{G}_\theta(L_k^\circ) \\
&= \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j [\bar{F}_\theta(L_k^\circ)]^{m-l+j} \\
&= \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \left(\exp \left[-\exp \left(\frac{L_k^\circ - \theta_1}{\theta_2} \right) \right] \right)^{m-l+j} \\
&= \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \\
&\times \left[\exp \left[-e^{w_1} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2) \right) \right] \right. \\
&\quad \left. \times \left(\exp \left(\frac{L_k - \hat{\theta}_1}{\hat{\theta}_2} \right) \right)^{W_2} \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2) \right)^{-1} \right]^{m-l+j} \\
&= \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \\
&\times \left[\exp \left[-W \left(\exp \left(\frac{L_k - \hat{\theta}_1}{\hat{\theta}_2} \right) \right)^{W_2} \right. \right. \\
&\quad \left. \left. \times \left(\sum_{i=1}^r \exp(S_i W_2) + (n-r) \exp(S_r W_2) \right)^{-1} \right] \right]^{m-l+j}.
\end{aligned}$$

It follows from (54) that

$$L_k^\circ = \arg \left(\sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{v(\mathbf{s}^{(r)})} \right. \\ \left. \times \frac{w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2)}{\left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right)^r} dw_2 = 1 - \alpha \right). \quad (55)$$

Assuming that

$$\exp\left(\frac{L_k^\circ - \hat{\theta}_1}{\hat{\theta}_2}\right) = \eta_{L_k^\circ}, \quad (56)$$

we have (51). This completes the proof. \square

Theorem 6. Let $\underline{X}_1 \leq \dots \leq \underline{X}_r$ be the first r ordered observations of a random variable $\underline{X}(=\exp X)$ from a sample of size n from a two-parameter Weibull distribution defined by the probability density function (22). Then a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k^\circ(\underline{S})$, on future outcomes of the k th order statistic $\underline{Y}_k(=\exp Y_k)$ from a set of m future ordered observations $\underline{Y}_1 \leq \dots \leq \underline{Y}_m$ also from the distribution (22), which satisfies (6) is given by

$$\underline{L}_k^\circ = \eta_{\underline{L}_k}^{1/\delta} \hat{\beta} = \exp \underline{L}_k, \quad (57)$$

where $\eta_{\underline{L}_k}$ is a tolerance factor determined by

$$\eta_{\underline{L}_k} = \arg \left\{ \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{g(\mathbf{z}^{(r)})} \times \int_0^\infty \frac{v_2^{r-2} \prod_{i=1}^r z_i^{v_2}}{\left(\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right)^r} dv_2 = 1-\alpha \right\}. \quad (58)$$

Proof. The proof is similar to that of Theorem 5 and so it is omitted here.

Inference 3 (for the tolerance factors $\eta_{\underline{L}_k}$ and $\eta_{\underline{U}_k}$). It follows from (56) and (57) that

$$\eta_{\underline{L}_k} = \left(\frac{\underline{L}_k}{\hat{\beta}} \right)^\delta = \left(\frac{\exp \underline{L}_k}{\exp \hat{\theta}_1} \right)^{1/\hat{\theta}_2} = \exp \left(\frac{\underline{L}_k - \hat{\theta}_1}{\hat{\theta}_2} \right) = \eta_{\underline{L}_k}. \quad (59)$$

B. Constructing Upper Statistical Tolerance Limit with Expected $(1-\alpha)$ -Confidence

Theorem 7. Let $X_1 \leq \dots \leq X_r$ be the first r ordered observations of a random variable $X(=\ln \underline{X})$ from a sample of size n from an extreme-value distribution defined by the probability density function (8). Then an upper statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{U}_k^\circ \equiv \underline{U}_k^\circ(\underline{S})$, on future outcomes of the k th order statistic $\underline{Y}_k(=\ln Y_k)$ from a set of m future ordered observations $\underline{Y}_1 \leq \dots \leq \underline{Y}_m$ also from the distribution (8), which satisfies (7) is given by

$$\underline{U}_k^\circ = \hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{\underline{U}_k} = \ln \underline{U}_k, \quad (60)$$

where $\eta_{\underline{U}_k}$ is a tolerance factor determined by

$$\eta_{\underline{U}_k} = \arg \left\{ \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{v(\mathbf{s}^{(r)})} \times \int_0^\infty \frac{w_2^{r-2} \prod_{i=1}^r \exp(s_i w_2)}{\left(\sum_{i=1}^r \exp(s_i w_2) + (n-r) \exp(s_r w_2) \right)^r} dw_2 = \alpha \right\}. \quad (61)$$

Proof. The upper statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{U}_k^\circ \equiv \underline{U}_k^\circ(\underline{S})$, is obtained from a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k^\circ(\underline{S})$, by replacing $1-\alpha$ by α . This completes the proof.

Theorem 8. Let $\underline{X}_1 \leq \dots \leq \underline{X}_r$ be the first r ordered observations of a random variable $\underline{X}(=\exp X)$ from a sample of size n from a two-parameter Weibull distribution defined by the probability density function (22). Then an upper statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{U}_k^\circ \equiv \underline{U}_k^\circ(\underline{S})$, on future outcomes of the k th order statistic $\underline{Y}_k(=\exp Y_k)$ from a set of m future ordered observations $\underline{Y}_1 \leq \dots \leq \underline{Y}_m$ also from the distribution (22), which satisfies (7) is given by

$$\underline{U}_k^\circ = \eta_{\underline{U}_k}^{1/\delta} \hat{\beta} = \exp \underline{U}_k, \quad (62)$$

where $\eta_{\underline{U}_k}$ is a tolerance factor determined by

$$\eta_{\underline{U}_k} = \arg \left\{ \sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{g(\mathbf{z}^{(r)})} \times \int_0^\infty \frac{v_2^{r-2} \prod_{i=1}^r z_i^{v_2}}{\left(\sum_{i=1}^r z_i^{v_2} + (n-r)z_r^{v_2} \right)^r} dv_2 = \alpha \right\}. \quad (63)$$

Proof. The upper statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{U}_k^\circ \equiv \underline{U}_k^\circ(\underline{S})$, is obtained from a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k^\circ(\underline{S})$, by replacing $1-\alpha$ by α . This completes the proof.

Inference 4 (for the tolerance factors $\eta_{\underline{U}_k}$ and $\eta_{\underline{L}_k}$). It follows from (60) and (62) that

$$\eta_{\underline{U}_k} = \left(\frac{\underline{U}_k}{\hat{\beta}} \right)^\delta = \left(\frac{\exp \underline{U}_k}{\exp \hat{\theta}_1} \right)^{1/\hat{\theta}_2} = \exp \left(\frac{\underline{U}_k - \hat{\theta}_1}{\hat{\theta}_2} \right) = \eta_{\underline{U}_k}. \quad (64)$$

VI. Numerical Examples

A. Numerical Example 1

For the Weibull case, Lawless [17] discusses an example with 10 items, which are put on test simultaneously; the life test is terminated at the time of the fifth failure, whence $n=10$, $r=5$, in our notation here; $\underline{X}_1=50.5$ hours, $\underline{X}_2=71.3$, $\underline{X}_3=84.6$, $\underline{X}_4=98.7$, $\underline{X}_5=103.8$; the maximum likelihood estimates of δ and β are, respectively,

$$\hat{\delta} = \left[\frac{\left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} \ln \underline{x}_i + (n-r) \underline{x}_r^{\hat{\delta}} \ln \underline{x}_r \right)}{\left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} + (n-r) \underline{x}_r^{\hat{\delta}} \right)^{-1} - \frac{1}{r} \sum_{i=1}^r \ln \underline{x}_i} \right]^{-1} = 4.199, \quad (65)$$

$$\hat{\beta} = \left(r^{-1} \left(\sum_{i=1}^r \underline{x}_i^{\hat{\delta}} + (n-r) \underline{x}_r^{\hat{\delta}} \right) \right)^{1/\hat{\delta}} = 114.2796. \quad (66)$$

Based on these data, a lower 90% prediction limit (in terms of this paper, a lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, where $\alpha=0.1$) is to be constructed for the minimum of 40 independently, identically distributed lifetimes. Lawless [17] reports a conditional lower 90% prediction limit of 8.8 hours for this example. Based on a simulation of 50 000 samples, the lower prediction limit obtained by Mee and Kushary [18] is 8.73 hours.

Lower statistical tolerance limit with expected $(1-\alpha)$ -confidence. Taking $1-\alpha = 0.9$ and $k=1$, with $n=10$, $r=5$ and $m=40$, we have from (57) that the lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k^{\circ}(\underline{S})$, on the minimum (\underline{Y}_1) of independent lifetimes in a group of $m=40$ components which are to be put into service, is given by

$$\underline{L}_k^{\circ} = \eta_{\underline{L}_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k^{\circ} = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{\underline{L}_k}) = 8.7941146, \quad (67)$$

where, as it follows from (58) and (59), the tolerance factor $\eta_{\underline{L}_k}$ is given by

$$\eta_{\underline{L}_k} = \arg \left(\frac{1}{\mathcal{G}(\mathbf{z}^{(r)})} \int_0^{\infty} \frac{v_2^{r-2} \prod_{i=1}^r z_i^{v_2}}{\left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} + [\eta_{\underline{L}_k}^{v_2}]^m \right)^r} dv_2 = 1 - \alpha \right) \\ = \eta_{\underline{L}_k} = 2.105 / 10^5. \quad (68)$$

Statistical inference 1. It is easy to see that the conditional lower 90% prediction limit of 8.8 hours of Lawless [17] on the minimum (\underline{Y}_1) of independent lifetimes in a group of $m=40$ components, which are to be put into service, and the lower statistical tolerance limit of 8.7941146 hours with expected 0.9-confidence, which is

obtained in this paper by using the proposed technique, are practically the same.

Lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence. In the above case (if $\gamma=0.9$), it follows from (43) that the lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k(\underline{S})$, on the minimum (\underline{Y}_1) of independent lifetimes in a group of $m=40$ components which are to be put into service, is given by

$$\underline{L}_k = \eta_{\underline{L}_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{\underline{L}_k}) = 3.7, \quad (69)$$

where, as it follows from (44) and (45), the tolerance factor $\eta_{\underline{L}_k}$ is given by

$$\eta_{\underline{L}_k} = \arg \left(\frac{\int_0^{\infty} \frac{\ln(1-q_{1-\gamma})^{-1} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} \right)}{[\eta_{\underline{L}_k}]^{v_2}} \frac{1}{\Gamma(r)} w^{r-1} \exp(-w)}{\times \frac{1}{\mathcal{G}(\mathbf{z}^{(r)})} v_2^{r-2} \prod_{i=1}^r z_i^{v_2} \left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} \right)^{-r}} dw dv_2 = 1 - \alpha \right) \\ = \eta_{\underline{L}_k} = 5.5451 / 10^7, \quad (70)$$

$q_{1-\gamma} = 0.002631$ is a quantile of the beta distribution satisfying (38).

Statistical inference 2. Thus, the manufacturer has 90% assurance that no failures will occur in the proportion $\gamma=0.9$ or more of the population of $m=40$ components, which are to be put into service, before $\underline{L}_k = 3.7$ hours.

Others have computed approximate 90% prediction limits for a single future lifetime for this example. Fertig, Meyer and Mann [19] computed a lower prediction limit of 56.98 hours, using best linear invariant estimators and a Monte Carlo estimated percentile. Engelhardt and Bain [20] proposed two approximations; for this example they obtained 56.8 (via a procedure requiring iterative solution of a nonlinear equation) and 59.1 (via a simpler approximation). Based on a simulation of 50 000 samples, the 90 % lower prediction limit obtained by Mee and Kushary [18] for a single future observation is 56.6 hours.

Lower statistical tolerance limit with expected $(1-\alpha)$ -confidence. Taking $1-\alpha = 0.9$ and $k=m=1$, with $n=10$ and $r=5$, we have from (57) that the lower statistical tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k^{\circ}(\underline{S})$, on a single future observation is given by

$$\underline{L}_k^{\circ} = \eta_{\underline{L}_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k^{\circ} = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{\underline{L}_k}) = 56.641, \quad (71)$$

where, as it follows from (58) and (59), the tolerance factor $\eta_{\underline{L}_k}$ is given by

$$\eta_{\underline{L}_k} = \arg \left(\frac{1}{\mathcal{G}(\mathbf{z}^{(r)})} \int_0^{\infty} \frac{v_2^{r-2} \prod_{i=1}^r z_i^{v_2}}{\left(\sum_{i=1}^r z_i^{v_2} + (n-r) z_r^{v_2} + [\eta_{\underline{L}_k}^{v_2}]^m \right)^r} dv_2 = 1 - \alpha \right) \\ = \eta_{\underline{L}_k} = 0.052479. \quad (72)$$

Statistical inference 3. Based on a simulation of 50 000 samples, the 90 % lower prediction limit obtained by Mee and Kushary [18] for a single future lifetime is 56.6 hours, which is slightly smaller than 56.641 hours (see (71)). Engelhardt and Bain [20] proposed the first approximate lower prediction limit of 56.8 hours and Fertig, Meyer and Mann [19] computed the lower prediction limit of 56.98 hours for a single future lifetime, which are slightly larger than 56.641 hours (see (71)). The second approximate lower prediction limit of 59.1 hours proposed by Engelhardt and Bain [20] for a single future lifetime is larger than 56.641 hours (see (71)).

B. Numerical Example 2

Consider the following results given by Lieblein and Zelen [21] of test of endurance, in millions of revolutions, of $n=23$ ball bearings: 17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40. The maximum likelihood estimates of δ and β are, respectively,

$$\hat{\delta} = \left[\left(\sum_{i=1}^n x_i^{\hat{\delta}} \ln x_i \right) \left(\sum_{i=1}^n x_i^{\hat{\delta}} \right)^{-1} - \frac{1}{n} \sum_{i=1}^n \ln x_i \right]^{-1} = 2.102, \quad (73)$$

$$\hat{\beta} = \left(n^{-1} \left(\sum_{i=1}^n x_i^{\hat{\delta}} \right) \right)^{1/\hat{\delta}} = 81.878. \quad (74)$$

Numerous authors have used these data as illustrative of a sample from a two-parameter Weibull distribution. Using 20000 simulated samples of size $n=23$, Mee and Kushary [18] obtained a 90% lower prediction limit for the fifth failure out one hundred ball bearings equal to 10.11 million of revolutions, which is slightly smaller than the two approximate prediction limits 10.27 and 10.59 reported by Engelhardt and Bain [20].

Lower statistical tolerance limit with expected (1- α)-confidence. Taking $1-\alpha = 0.9$ and $k=5$, with $r=n=23$ and $m=100$, we have from (57) that the lower statistical tolerance limit with expected (1- α)-confidence, $\underline{L}_k \equiv \underline{L}_k^{\circ}(\underline{S})$, for a fifth failure out one hundred ball bearings is given by

$$\underline{L}_k = \eta_{L_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k^{\circ} = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k}) = 10.35206, \quad (75)$$

where, as it follows from (58) and (59), the tolerance factor η_{L_k} is given by

$$\eta_{L_k} = \arg \left(\frac{\sum_{l=0}^{k-1} \binom{m}{l} \sum_{j=0}^l \binom{l}{j} (-1)^j \frac{1}{\mathcal{G}(\mathbf{z}^{(n)})}}{v_2^{n-2} \prod_{i=1}^n z_i^{v_2}} \int_0^{\infty} \frac{1}{\left(\sum_{i=1}^n z_i^{v_2} + [\eta_{L_k}]^{v_2} \right)^n} dv_2 = 1 - \alpha \right) = \eta_{L_k} = 0.0129452. \quad (76)$$

Statistical inference 4. Thus, it follows from (75) that the lower statistical tolerance limit with expected confidence

of 0.9 ($\underline{L}_k^{\circ} = 10.35206$ million of revolutions) is between the two approximate prediction limits 10.27 and 10.59 reported by Engelhardt and Bain [20].

Lower statistical tolerance limit with expected (1- α)-confidence. Taking $1-\alpha = 0.9$ and $k=1$, with $r=n=23$ and $m=100$, we have from (57) that the lower statistical tolerance limit with expected (1- α)-confidence, $\underline{L}_k \equiv \underline{L}_k^{\circ}(\underline{S})$, for a first failure out one hundred ball bearings, is given by

$$\underline{L}_k = \eta_{L_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k^{\circ} = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k}) = 2.083, \quad (77)$$

where, as it follows from (58) and (59), the tolerance factor η_{L_k} is given by

$$\eta_{L_k} = \arg \left(\frac{1}{\mathcal{G}(\mathbf{z}^{(n)})} \int_0^{\infty} \frac{v_2^{n-2} \prod_{i=1}^n z_i^{v_2}}{\left(\sum_{i=1}^n z_i^{v_2} + [\eta_{L_k}]^{v_2} \right)^n} dv_2 = 1 - \alpha \right) = \eta_{L_k} = 0.00044503. \quad (78)$$

Statistical inference 5. Lawless [17] obtained for this example (via conditional approach in terms of a extreme-value (Gumbel) distribution) the lower 90% prediction limit of 2.08 million of revolutions for a first failure out one hundred ball bearings, which is slightly smaller than 2.083 (see (77)).

C. Numerical Example 3

Consider the data in an example discussed by Mann and Saunders [22]. They regard the data coming from the Weibull distribution as the results of full scale fatigue tests on a particular type of component. The data are for a complete sample of size $n=3$, with observations $X_1 = 45.952$, $X_2 = 54.143$, and $X_3 = 65.440$, results being expressed here in number of thousands of cycles. On the basis of these data it is wished to obtain the lower statistical tolerance limit with expected (1- α)-confidence for the minimum (Y_1) of independent lifetimes in a group of $m=500$ components which are to be put into service.

Lower statistical tolerance limit with expected (1- α)-confidence. The maximum likelihood estimates of the unknown parameters δ and β , computed on the basis of (X_1, X_2, X_3) , are $\hat{\delta} = 7.726$ and $\hat{\beta} = 58.706$, respectively. Taking $1-\alpha = 0.8$ and $k=1$, with $r=n=3$ and $m=500$, we have from (57) that the lower statistical tolerance limit with expected (1- α)-confidence, $\underline{L}_k \equiv \underline{L}_k^{\circ}(\underline{S})$, on the minimum (Y_1) of independent lifetimes in a group of $m=500$ components which are to be put into service, is given by

$$\underline{L}_k = \eta_{L_k}^{1/\hat{\delta}} \hat{\beta} = \exp L_k^{\circ} = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k}) = 5.527411, \quad (79)$$

where, as it follows from (58) and (59), the tolerance factor η_{L_k} is given by

$$\eta_{L_k} = \arg \left(\frac{1}{g(\mathbf{z}^{(n)})} \int_0^\infty \frac{v_2^{n-2} \prod_{i=1}^n z_i^{v_2}}{\left(\sum_{i=1}^n z_i^{v_2} + \left[\eta_{L_k} \right]^{v_2} m \right)^n} dv_2 = 1 - \alpha \right)$$

$$= \eta_{L_k} = 1.18 / 10^8. \quad (80)$$

Statistical inference 6. Lawless [17] obtained for this example (via conditional approach in terms of the extreme-value (Gumbel) distribution) the lower 80% prediction limit of 5.623, which is slightly larger than 5.527411 (see (79)). The resulting lower 80% prediction limit of Mee and Kshary [18] for this example (obtained via simulation of 100 000 samples) was 5.225, which is slightly smaller than 5.527411 (see (79)). The Mann and Saunders [22] result for this example was only 0.766. All results are expressed here in the number of thousands of cycles.

Lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence. Taking $\gamma=0.8$, $1-\alpha=0.8$ and $k=1$, with $r=n=3$ and $m=500$, we have from (43) that a lower statistical γ -content tolerance limit with expected $(1-\alpha)$ -confidence, $\underline{L}_k \equiv \underline{L}_k(\underline{S})$, on the minimum (Y_1) of independent lifetimes in a group of $m=500$ components which are to be put into service, is

$$\underline{L}_k = \eta_{L_k}^{1/\delta} \hat{\beta} = \exp L_k = \exp(\hat{\theta}_1 + \hat{\theta}_2 \ln \eta_{L_k}) = 4.082282, \quad (81)$$

where, as it follows from (44) and (45), the tolerance factor η_{L_k} is given by

$$\eta_{L_k} = \arg \left(\int_0^\infty \int_0^\infty \frac{\ln(1-q_{1-\gamma})^{-1} \sum_{i=1}^n z_i^{v_2}}{[\eta_{L_k}]^{v_2}} \frac{1}{\Gamma(n)} w^{n-1} \exp(-w) \right. \\ \left. \times \frac{1}{g(\mathbf{z}^{(n)})} v_2^{n-2} \prod_{i=1}^n z_i^{v_2} \left(\sum_{i=1}^n z_i^{v_2} \right)^{-n} dw dv_2 = 1 - \alpha \right)$$

$$= 1.135 / 10^9, \quad (82)$$

$q_{1-\gamma}$ is a quantile of the beta distribution satisfying (38).

Statistical inference 7. Thus, the manufacturer has 80% assurance that no failures will occur in the proportion $\gamma=0.8$ or more of the population of $m=500$ components, which are to be put into service, before $\underline{L}_k = 4.082282$ thousands of cycles.

VII. New Technique of Invariant Statistical Embedding and Pivotal Quantity Averaging to Construct Effective Statistical Decisions under Parametric Uncertainty

Analytical formulas for the tolerance limits are available in the scientific literature for only simple cases, for example, for the upper or lower tolerance limit for a univariate normal population. Thus it becomes necessary to use new methods

in order to derive exact statistical tolerance limits for many populations. The proposed, in this paper, technique of Invariant Statistical Embedding and Pivotal Quantity Averaging (ISE & PQA) for constructing exact tolerance limits to predict future outcomes coming from log-location-scale distributions under parametric uncertainty represents a novelty in the theory of statistical decisions. This simple and computationally attractive statistical method is based on the constructive use of the invariance principle in mathematical statistics. It allows one to improve the decision-making process under parametric uncertainty by removing unknown parameters from the problem and using the past data as completely as possible. The technique of ISE & PQA includes the following 3 steps:

Step 1. Invariant embedding of a sample statistic in the decision criterion to construct a *pivotal quantity* (or simply a *pivot*) to isolate the unknown parameter (where the *pivot's probability distribution does not depend on the unknown parameter*);

Step 2. The decision criterion is averaged over the pivotal quantities to exclude the unknown parameters from the problem;

Step 3. When the unknown parameters are excluded from the decision criterion, then it can be found an effective statistical decision rule.

A. Numerical Example 4

Consider, for example, the problem of estimating a quantile u of an exponential distribution on the basis of a random sample X_1, \dots, X_n of size $n \geq 2$. The exponential distribution is often used for length of life data. The exponential probability density function (pdf) is given by

$$f_\theta(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \quad \theta > 0. \quad (83)$$

The cumulative distribution function (cdf) is given by

$$F_\theta(x) = 1 - \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \quad \theta > 0. \quad (84)$$

Quantile estimation, particularly for the exponential distribution, is important in reliability theory, life testing, and so on. Also, in statistical decision theory it is of interest to find out if the best equivariant estimator or the maximum likelihood estimator of quantile is admissible.

Thus, the problem is to estimate the p^{th} quantile $u = \eta\theta$ of the exponential distribution, where $0 < \eta = -\ln(1-p)$; $0 < p < 1$. The loss function is taken as

$$L(\theta, d) = (F_\theta(d) - p)^2, \quad (85)$$

where d is an estimator (decision rule) for estimating the quantile u . We evaluate the performance of an estimator for quantile with the help of the risk function (decision criterion).

$$R(\theta, d) = E_\theta \{L(\theta, d)\}. \quad (86)$$

Assuming that the parameter θ is unknown, we find the maximum likelihood estimator (MLE) of θ given by

$$\hat{\theta} = \sum_{i=1}^n X_i / n. \quad (87)$$

It is known that

$$\hat{\theta} \sim \varphi_{\theta}(\hat{\theta}) = \frac{n^n}{\Gamma(n)\theta^n} \hat{\theta}^{n-1} \exp\left(-\frac{n\hat{\theta}}{\theta}\right), \quad s > 0, \quad \theta > 0, \quad (88)$$

where

$$V = \hat{\theta}/\theta \quad (89)$$

represents a pivotal quantity with the probability density function

$$\varphi(v) = \frac{n^n}{\Gamma(n)} v^{n-1} \exp(-nv), \quad v > 0. \quad (90)$$

To solve the above problem, the technique of invariant statistical embedding and pivotal quantity averaging, proposed in this paper, can be used.

Using the technique of ISE & PQA, we have the following:

Step 1. Invariant embedding of MLE $\hat{\theta}$ in the decision criterion to construct the pivotal quantity V :

$$\begin{aligned} R(\theta, d) &= E_{\theta} \{L(\theta, d)\} = E_{\theta} \{(F_{\theta}(d) - p)^2\} \\ &= E_{\theta} \left\{ \left(1 - \exp\left(-\frac{d}{\theta}\right) - p \right)^2 \right\} = E_{\theta} \left\{ \left(1 - p - \exp\left(-\frac{d}{\theta}\right) \right)^2 \right\} \\ &= E_{\theta} \left\{ (1-p)^2 - 2(1-p) \exp\left(-\frac{d}{\theta}\right) + \exp\left(-\frac{2d}{\theta}\right) \right\} \\ &= E_{\theta} \left\{ (1-p)^2 - 2(1-p) \exp\left(-\frac{d}{\theta} \frac{\hat{\theta}}{\hat{\theta}}\right) + \exp\left(-\frac{2d}{\theta} \frac{\hat{\theta}}{\hat{\theta}}\right) \right\} \\ &= E \{(1-p)^2 - 2(1-p) \exp(-\mathcal{G}V) + \exp(-2\mathcal{G}V)\}, \quad (91) \end{aligned}$$

where

$$\mathcal{G} = d/\hat{\theta}. \quad (92)$$

Step 2. Averaging of decision criterion over the pivotal quantity V :

$$\begin{aligned} R(\theta, d) &= E \{(1-p)^2 - 2(1-p) \exp(-\mathcal{G}V) + \exp(-2\mathcal{G}V)\} \\ &= \int_0^{\infty} [(1-p)^2 - 2(1-p) \exp(-\mathcal{G}v) + \exp(-2\mathcal{G}v)] \varphi(v) dv \\ &= \int_0^{\infty} [(1-p)^2 - 2(1-p) \exp(-\mathcal{G}v) + \exp(-2\mathcal{G}v)] \\ &\quad \times \frac{n^n}{\Gamma(n)} v^{n-1} \exp(-nv) dv \\ &= (1-p)^2 - 2(1-p) \frac{n^n}{(n+\mathcal{G})^n} + \frac{n^n}{(n+2\mathcal{G})^n} \\ &= (1-p)^2 - 2(1-p) \frac{1}{(1+\mathcal{G}/n)^n} + \frac{1}{(1+2\mathcal{G}/n)^n}. \quad (93) \end{aligned}$$

Step 3. Process of finding the optimal statistical decision rule:

If $p=0.8$, $n=2$ and $\hat{\theta}=10$, it can be shown that

$$\mathcal{G}^* = \arg \min R(\theta, d)$$

$$\begin{aligned} &= \arg \min \left((1-p)^2 - 2(1-p) \frac{1}{(1+\mathcal{G}/n)^n} \right. \\ &\quad \left. + \frac{1}{(1+2\mathcal{G}/n)^n} \right) = 4.89598, \quad (94) \end{aligned}$$

The optimal estimator (statistical decision rule) d for estimating the quantile u is given by

$$d = \mathcal{G}^* \hat{\theta} = 4.89598 \times 10 = 48.9598, \quad (95)$$

and the risk function is equal to

$$R(\theta, d) = E_{\theta} \{L(\theta, d)\} = 0.035121. \quad (96)$$

For comparison, the maximum likelihood estimator d_{ML} for estimating the quantile u is given by

$$\begin{aligned} d_{ML} &= \eta \hat{\theta} = -\ln(1-p) \hat{\theta} \\ &= 1.609438 \times 10 = 16.09438, \quad (97) \end{aligned}$$

and the risk function is equal to

$$R(\theta, d_{ML}) = E_{\theta} \{L(\theta, d_{ML})\} = 0.064049. \quad (98)$$

The index of improvement percentage of accuracy in estimating the quantile u by d as compared with accuracy of estimating the quantile u by d_{ML} is given by

$$\begin{aligned} I_{\text{imp.per.}}(d, d_{ML}) &= \frac{R(\theta, d_{ML}) - R(\theta, d)}{R(\theta, d_{ML})} 100\% \\ &= \frac{0.064049 - 0.035121}{0.064049} 100\% = 45.16531\%. \quad (99) \end{aligned}$$

B. Characterization of Uniformly Non-Dominated Statistical Decision Rules

A decision rule d is said to be uniformly non-dominated if there is no decision rule uniformly better than d . The conditions that a decision rule must satisfy in order that it might be uniformly non-dominated are given by the following theorem.

Theorem 9 (Uniformly non-dominated decision rule). Let $(\xi_{\tau}(\theta); \tau=1, 2, \dots)$ be a sequence of the prior distributions on the parameter space Θ . Suppose that $(d_{\tau}; \tau=1, 2, \dots)$ and $(R^*(\xi_{\tau}(\theta), d_{\tau}); \tau=1, 2, \dots)$ are the sequences of Bayes decision rules and posterior risks, respectively. If there exists a statistical decision rule d such that its risk function $R(\theta, d)$, $\theta \in \Theta$, satisfies the relationship

$$\lim_{\tau \rightarrow \infty} [R^*(\xi_{\tau}(\theta), d) - R^*(\xi_{\tau}(\theta), d_{\tau})] = 0, \quad (100)$$

where a posterior risk (obtained through the posterior pdf $\xi_{\tau}^*(\theta)$ of θ is given by

$$R^*(\xi_{\tau}(\theta), d) = \int_{\Theta} R(\theta, d) \xi_{\tau}^*(\theta) d\theta, \quad (101)$$

then d is a uniformly non-dominated decision rule.

Proof. Suppose d is uniformly dominated. Then there exists a decision rule d^* such that $R(\theta, d^*) < R(\theta, d)$ for all $\theta \in \Theta$. Let

$$\varepsilon = \inf_{\theta \in \Theta} [R(\theta, d) - R(\theta, d^*)] > 0. \quad (102)$$

Then

$$R^*(\xi_\tau(\theta), d) - R^*(\xi_\tau(\theta), d^*) \geq \varepsilon. \quad (103)$$

Simultaneously,

$$R^*(\xi_\tau(\theta), d^*) - R^*(\xi_\tau(\theta), d_\tau) \geq 0, \quad (104)$$

$\tau = 1, 2, \dots$, and

$$\lim_{\tau \rightarrow \infty} [R^*(\xi_\tau(\theta), d^*) - R^*(\xi_\tau(\theta), d_\tau)] \geq 0. \quad (105)$$

On the other hand,

$$\begin{aligned} & R^*(\xi_\tau(\theta), d^*) - R^*(\xi_\tau(\theta), d_\tau) \\ &= [R^*(\xi_\tau(\theta), d) - R^*(\xi_\tau(\theta), d_\tau)] \\ & \quad - [R^*(\xi_\tau(\theta), d) - R^*(\xi_\tau(\theta), d^*)] \\ & \leq [R^*(\xi_\tau, d) - R^*(\xi_\tau, d_\tau)] - \varepsilon \end{aligned} \quad (106)$$

and

$$\lim_{\tau \rightarrow \infty} [R^*(\xi_\tau, d^*) - R^*(\xi_\tau, d_\tau)] < 0. \quad (107)$$

This contradiction proves that d is a uniformly non-dominated decision rule.

C. Bayes Estimator of the p^{th} Quantile

Let X_1, \dots, X_n be identically and independently distributed random variables taken from one-parameter exponential distribution (83). The likelihood function is given by

$$\begin{aligned} L(X_1, \dots, X_n | \theta) &= \prod_{i=1}^n \frac{\exp(-X_i / \theta)}{\theta} \\ &= \frac{\exp(-S / \theta)}{\theta^n}, \end{aligned} \quad (108)$$

where

$$S = \sum_{i=1}^n X_i. \quad (109)$$

Considering the inverted gamma prior, the prior pdf (probability density function) of θ is given by

$$\xi_{a,b}(\theta) = \frac{a^b}{\Gamma(b)} \frac{\exp(-a/\theta)}{\theta^{b+1}}, \quad a > 0, \quad b > 0, \quad \theta > 0, \quad (110)$$

where

$$a = b = \frac{1}{\tau} \quad (111)$$

and

$$\xi_\tau(\theta) = \frac{\left(\frac{1}{\tau}\right)^{1/\tau} \exp\left(-\frac{1}{\tau\theta}\right)}{\Gamma\left(\frac{1}{\tau}\right) \theta^{1/\tau+1}}. \quad (112)$$

The joint pdf of X_1, \dots, X_n and θ is given by

$$\begin{aligned} f_{a,b}(x_1, \dots, x_n, \theta) &= \frac{a^b}{\Gamma(b)} \frac{\exp(-[s+a]/\theta)}{\theta^{n+b+1}}, \\ a > 0, \quad b > 0, \quad \theta > 0. \end{aligned} \quad (113)$$

The marginal pdf of X_1, \dots, X_n is given by

$$\begin{aligned} f_{a,b}(x_1, \dots, x_n) &= \int_0^\infty f_{a,b}(x_1, \dots, x_n, \theta) d\theta \\ &= \int_0^\infty \frac{a^b}{\Gamma(b)} \frac{\exp(-[s+a]/\theta)}{\theta^{n+b+1}} d\theta = \frac{a^b}{\Gamma(b)} \frac{\Gamma(n+b)}{(s+a)^{n+b}} \end{aligned} \quad (114)$$

Now the posterior pdf of θ is given by

$$\begin{aligned} f_{a,b}(\theta | x_1, \dots, x_n) &= \frac{f_{a,b}(x_1, \dots, x_n, \theta)}{f_{a,b}(x_1, \dots, x_n)} \\ &= \frac{(s+a)^{n+b}}{\Gamma(n+b)} \frac{1}{\theta^{n+b+1}} \exp\left(-\frac{s+a}{\theta}\right) \\ &= \frac{(n\hat{\theta}+a)^{n+b}}{\Gamma(n+b)} \frac{1}{\theta^{n+b+1}} \exp\left(-\frac{n\hat{\theta}+a}{\theta}\right) \\ &= \frac{(n+a/\hat{\theta})^{n+b}}{\Gamma(n+b)} \frac{\hat{\theta}^{n+b}}{\theta^{n+b+1}} \exp\left(-\frac{(n+a/\hat{\theta})\hat{\theta}}{\theta}\right), \\ a > 0, \quad b > 0, \quad \theta > 0. \end{aligned} \quad (115)$$

It follows from (115) that

$$\begin{aligned} \xi_\tau^*(\theta) &= \frac{[n+(1/\tau)/\hat{\theta}]^{n+1/\tau}}{\Gamma(n+1/\tau)} \frac{\hat{\theta}^{n+1/\tau}}{\theta^{n+1/\tau+1}} \\ &\times \exp\left(-\frac{[n+(1/\tau)/\hat{\theta}]\hat{\theta}}{\theta}\right), \quad \theta > 0. \end{aligned} \quad (116)$$

$$\begin{aligned} V &= \frac{\hat{\theta}}{\theta} \sim \varphi_{a,b}^*(v) = \frac{(n+a/\hat{\theta})^{n+b}}{\Gamma(n+b)} \\ &\times v^{n+b-1} \exp(-(n+a/\hat{\theta})v), \quad v > 0. \end{aligned} \quad (117)$$

Thus, it can be shown that

$$\begin{aligned} & \int_0^\infty R(\theta, d) f_{a,b}(\theta | x_1, \dots, x_n) d\theta \\ &= \int_0^\infty [(1-p)^2 - 2(1-p) \exp(-\theta v) + \exp(-2\theta v)] \varphi_{a,b}^*(v) dv \end{aligned}$$

$$\begin{aligned}
 &= (1-p)^2 - 2(1-p) \frac{(n+a/\hat{\theta})^{n+b}}{[(n+a/\hat{\theta})+g]^{n+b}} + \frac{(n+a/\hat{\theta})^{n+b}}{[(n+a/\hat{\theta})+2g]^{n+b}} \\
 &= (1-p)^2 - 2(1-p) \frac{1}{[1+g/(n+a/\hat{\theta})]^{n+b}} \\
 &\quad + \frac{1}{[1+2g/(n+a/\hat{\theta})]^{n+b}}. \quad (118)
 \end{aligned}$$

It follows from (118) that

$$\begin{aligned}
 R^*(\xi_r(\theta), d) &= (1-p)^2 - 2(1-p) \frac{1}{(1+g/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}} \\
 &\quad + \frac{1}{(1+2g/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}} \quad (119)
 \end{aligned}$$

and

$$\begin{aligned}
 R^*(\xi_r(\theta), d_r) &= (1-p)^2 - 2(1-p) \frac{1}{(1+g_r/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}} \\
 &\quad + \frac{1}{(1+2g_r/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}}, \quad (120)
 \end{aligned}$$

where

$$g_r = \frac{d_r}{\hat{\theta}}. \quad (121)$$

It follows from (119) and (120) that

$$\begin{aligned}
 g_r^* &= \arg \min R^*(\xi_r, d_r) \\
 &= \arg \min \left((1-p)^2 - 2(1-p) \frac{1}{(1+g_r/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}} \right. \\
 &\quad \left. + \frac{1}{(1+2g_r/[n+(1/\tau)/\hat{\theta}])^{n+1/\tau}} \right), \quad (122)
 \end{aligned}$$

$$d_r = g_r^* \hat{\theta}, \quad \lim_{\tau \rightarrow \infty} g_r^* = g^*, \quad \text{and} \quad \lim_{\tau \rightarrow \infty} d_r = g^* \hat{\theta} = d, \quad (123)$$

It follows from (122) and (123) that

$$\lim_{\tau \rightarrow \infty} [R^*(\xi_r, d) - R^*(\xi_r, d_r)] = 0. \quad (124)$$

Statistical inference 8. Thus, the optimal estimator (decision rule) d for estimating the quantile u , obtained through the proposed, in this paper, novel technique of invariant statistical embedding and pivotal quantity averaging (ISE & PQA), is uniformly non-dominated.

VIII. Future Research Directions

Predictive inferences (via tolerance limits) for future outcomes on the basis of the past and present knowledge represent a fundamental problem of statistics, arising in many contexts and producing varied solutions. In this paper, new-sample prediction (via tolerance limits) based on a

previous sample is considered (i.e., when for predicting the future outcomes in a new sample there are available the observed data only from a previous sample). It is interesting to consider within-sample prediction (via tolerance limits) based on the early data from a current experiment (i.e., when for predicting the future outcomes in a sample there are available the early data only from that sample), and new-within-sample prediction (via tolerance limits) based on both the early data from that sample and the data from a previous sample (i.e., when for predicting the future outcomes in a new sample there are available both the early data from that sample and the data from a previous sample), where it is assumed that only the functional form of the underlying distributions is specified, but some or all of its parameters are unspecified.

A. Applications of Exact Statistical Tolerance Limits to Predict Future Random Quantities for Prognostics and Health Management of Complex Systems

Prognostics and Health Management (PHM) is a set of means, approaches, methods and tools that allows monitoring and tracking the health state of a system in order to detect, diagnose and predict its failures. Topics of interest include, but are not limited to:

(1) Prognostics and Health Management in Cyber-Physical Systems; (2) Fault detection, diagnostics and prognostics (data-driven, model-based and hybrid methods); (3) Remaining Useful Life (RUL) computation and prediction; (4) Post-diagnostics and post-prognostics decision; (5) Condition monitoring and sensors placement and optimization; (6) Feature extraction and selection, Health Indicators construction; (7) Modelling and simulation of interdependent failure mechanisms; (8) Advanced computation and simulation methods; (9) Industrial applications.

Prognostics and Health Management (PHM) aims at development and publishing original scientific contributions and industrial applications dealing with the topics covered by PHM in the areas of condition monitoring, fault detection, fault diagnostics, fault prognostics and decision support. The results obtained are then exploited to take appropriate decisions to increase the system's availability, reliability and security while reducing its maintenance costs.

B. Development of Novel Technologies of Prognostics and Health Management (PHM) for l-out-of-m Systems on the Basis of the Exact Statistical Tolerance Limits

Many technical systems or subsystems have l -out-of- m structure. These so-called l -out-of- m systems consist of m components of the same kind. The entire system is working if at least l of its m components are operating. It fails if $m-l+1$ or more components fail. Hence, a l -out-of- m system breaks down at the time of the $(m-l+1)$ th component failure. Since all components start working at the same time,

this approach leads to a kind of redundancy called active redundancy of $m-l$ components. Important particular cases of l -out-of- m systems are parallel and series systems corresponding to $l = 1$ and $l = m$, respectively. Practical examples of l -out-of- m systems are, e.g., an aircraft with four engines which will not crash if at least two out of its four engines remain functioning, or a satellite which will have enough power to send signals if not more than four out of its ten batteries are discharged.

Prediction problems come up naturally in several real life situations. For example, they can be broadly classified under two categories: (i) the variable to be predicted comes from the same experiment or sample so that it may be correlated with the observed data, (ii) it comes from an independent future experiment. Both of these situations do arise in the context of reliability and life testing. Formally the problems in the first category are known as one-sample (or within-sample) problems and those in the latter constitute two-sample (or new-sample) problems as well as multi-sample problems.

Suppose a system consists of m components and fails whenever l of these components fail. Such a system is referred to as l -out-of- m system. Suppose our observations consist of the first k failure times, and the goal is to predict the failure time of the system. Assuming that the components' life lengths are identically and independently distributed, we have a prediction problem involving a Type II censored sample, and it falls into category (i). Formally, the problem is to predict Y_l having observed $(Y_1 \leq \dots \leq Y_k)$, $k < l$. A popular choice for the lifelength of the components is the exponential distribution. Thus, one can think of a point predictor or an interval predictor for the lifelength of the system as one-sample (or within-sample) problem.

Consider the third situation where a manufacturer of certain equipment is interested in setting up a warranty for the equipment in a lot being sent out to the market. Using the information based on a small sample, possibly censored, the goal is to predict and set a lower prediction limit for the weakest item in a future sample. Typical assumption here is that the two samples are independent. This falls into category (ii), and we call it a two-sample (or new-sample) problem. One may be interested in the lifelength of the l -th weakest item and the average lifelength simultaneously. Then our focus will be on a prediction region for the two variables of interest. While several researches have concentrated on the above problems there are some papers dealing with special prediction situations involving several independent random samples. These belong to category (ii) and can be taken into account.

In reliability theory, the lifetime of a l -out-of- m system is usually described by the $(m-l+1)$ th order statistic Y_{m-l+1} from the sample $Y_1 \leq \dots \leq Y_m$, where the random variable Y_i represents the lifetime or failure time of the i th component of the system, $1 \leq i \leq m$. In the conventional modelling of these structures, the component lifetimes are supposed to be independent and identically distributed random variables. *Translating this approach back into the technical sphere, it reflects the assumption that the failure of any component does not affect the remaining ones. However, the supposition that the breakdown of some component will have no impact on the system parts at work will generally not be fulfilled in practice. In some systems, a*

component failure will more or less strongly influence the remaining parts of the system. For example, the breakdown of an aircraft's engine will increase the load put on the remaining engines such their lifetimes tend to be shorter. Thus, a more flexible model, which is therefore more applicable to practical situations, must take some dependence among the system components into account.

Prognostics and Health Management (PHM) offers several benefits for predictive maintenance. It predicts the future behavior of l -out-of- m system as well as its Remaining Useful Life (RUL). This RUL is used to plan the maintenance operation to avoid the failure, the stop time and optimize the cost of the maintenance and failure. However, with the development of the industry the assets are nowadays distributed, this is why the PHM needs to be developed using the new Information Technologies. In this paper, we propose a PHM solution based on the exact statistical prediction and tolerance limits on future random quantities (for example, future order statistics coming from Log-Location-Scale Distributions). These statistical limits will be constructed via the Pivotal Quantity Averaging Approach (PQAA), which represents the conceptually simple, efficient and useful method for constructing exact statistical prediction and tolerance limits on future outcomes under parametric uncertainty of underlying models.

Nechval et al. [23-24] discuss some of the problems described in this section.

C. ***Development of Novel Technologies of Prognostics and Health Management (PHM) for Fatigued Structures on the Basis of the Exact Statistical Tolerance Limits***

From an engineering standpoint the fatigue life of a fatigued structure consists of two periods: (i) crack initiation period, which starts with the first load cycle and ends when a technically detectable crack is presented, and (ii) crack propagation period, which starts with a technically detectable crack and ends when the remaining cross section can no longer withstand the loads applied and fails statically. The main aim of this paper is to present more accurate innovative stochastic fatigue model for adaptive planning inspections of fatigued structures in damage tolerance situations via observations of crack growth process during a crack propagation period. A new crack growth equation is based on this model. It is attractively simple and easy to apply in practice for effective in-service inspection planning (with decreasing intervals between sequential inspections as alternative to constant intervals often used in practice for convenience in operation). During the period of crack propagation (when the damage tolerance situation is used), the proposed crack growth equation, based on the innovative model, allows one to construct more accurate and effective reliability-based inspection strategy in this case.

Prognostics and Health Management (PHM) is a technology to enhance the effective reliability and availability of a fatigued structure in its life cycle conditions by detection of current and approaching failures and by providing for mitigation of the structure risks.

Prognostics is the real-time enhancement of reliability and availability and the prediction of the remaining useful

life of the structure by assessing the extent of deviation or degradation of the monitored parameters of the structure from its expected normal operating conditions. Prognostics can yield an advance warning of impending failure in a structure, thereby enabling more efficient and effective maintenance and corrective actions. Prognostics help in preventing catastrophic failures and can reduce unscheduled maintenance expenses. The outputs of a prognostic assessment of a structure are the failure risk, time to failure, remaining useful life, and a prognostic distance within which maintenance and repair actions can be planned, in order to minimize their impact on system availability.

Health Management is the process of using diagnostic and prognostic information to intelligently manage the use and maintenance of a system. The ultimate result is increased effective reliability, availability, and safety with reduced logistics and support costs.

Prognostics and Health Management (PHM) can be used to solve the service problems of the following important engineering structures: (1) Transportation Systems and Vehicles – aircraft, space vehicles, trains, ships; (2) Civil Structures – bridges, dams, tunnels; (3) Power Generation – nuclear, fossil fuel and hydroelectric plants; (4) High-Value Manufactured Products – launch systems, satellites, semiconductor and electronic equipment; (5) Industrial Equipment – oil and gas exploration, production and processing equipment, chemical process facilities, pulp and paper.

Nechval et al. [25-26] discuss some of the problems described in this section.

D. Development of Novel Technologies of Prognostics and Health Management (PHM) for a New Product in Systems of Lifetime Testing on the Basis of the Exact Statistical Tolerance Limits

A new product lifetime testing represents the problem that can be stated as follows. A new product is submitted for lifetime testing. The product will be accepted if a random sample of n items shows r or fewer failures in performance testing. We want to know whether to stop the test before it is completed if the results of the early observations are unfavourable. A suitable stopping decision saves the cost of the waiting time for completion.

On the other hand, an incorrect stopping decision causes an unnecessary design change and a complete rerun of the test. It is assumed that the redesign would improve the product to such an extent that it would definitely be accepted in a new lifetime testing. The paper presents a stopping rule based on the statistical estimation of total costs involved in the decision to continue beyond an early failure. Sampling is both expensive and time consuming. Hence, there are situations where it is more efficient to take samples sequentially, as opposed to all at one time, and to define a stopping rule to terminate the sampling process.

At the planning stage of a statistical investigation the question of sample size (n) is critical. For such an important issue, there is a surprisingly small amount of published

literature. Engineers who conduct reliability tests need to choose the sample size when designing a test plan. The model parameters and quantiles are the typical quantities of interest. The large-sample procedure relies on the property that the distribution of the t -like quantities is close to the standard normal in large samples. To estimate these quantities the maximum likelihood method is often used. The large-sample procedure to obtain the sample size relies on the property that the distribution of the above quantities is close to standard normal in large samples. The normal approximation is only first order accurate in general. When sample size is not large enough or when there is censoring, the normal approximation is not an accurate way to obtain the confidence intervals. Thus sample size determined by such procedure is dubious. Therefore, it may be considered the problem of constructing a test which minimizes the maximum expected sample size under some constraints. Stopping rules in sample testing can save substantial time and resources, when the case is clear-cut.

Prognostics and Health Management (PHM) is essential in guaranteeing the safe, efficient, and correct operation of complex of detection, isolation and identification of faults; and prognosis, which consists of prediction of the remaining useful life (RUL) of components, subsystems, or systems, constitutes system health monitoring. PHM aims to provide users with an integrated view of the health state of equipment or overall system.

Nechval et al. [27] discusses some of the problems described in this section.

IX. Conclusion

In this article, we construct the following one-sided statistical tolerance limits: i) one-sided statistical tolerance limit that covers at least $100\gamma\%$ of the measurements with expected $100(1-\alpha)\%$ confidence, ii) one-sided statistical tolerance limit determined so that the expected proportion of the measurements covered by this limit is $(1-\alpha)$.

Tolerance limits have important role in application of statistical methods in technical practice, especially in statistical quality control. Inherent in every phase of industrial quality control is the problem of comparing some quality characteristic or measurement of a finished product against given specifications. Sometimes the specifications, or *tolerance limits*, are so stated by the customer or by design engineer that any appreciable departure will make the product unusable. There remains, however, the problem of producing the part so that an acceptably high proportion of units will fall within tolerance limits specified for the given quality characteristic. Also, if a product is made without prior specifications, or if modifications are made, it is desirable to know within what limits the process can hold a quality characteristics a reasonably high percentage of the time. We thus speak of natural tolerance limits; that is, we let the process establish its own limits which, according the experience, can be met in actual practice.

The new analytical technique proposed in this article represents the conceptually simple, efficient and useful method for constructing exact statistical tolerance limits on future outcomes under parametric uncertainty of underlying models. It does not in need to make any assumption concerning the statistical equation for the tolerance limit. This technique, using the experimental complete or type II

censored data, is based on the idea of invariant embedding of a sample statistic in the underlying model to construct pivotal quantities and to eliminate the unknown parameters from the problem via pivotal quantity averaging. In this case, the exact statistical tolerance limits (under parametric uncertainty of underlying models) on future outcomes (say, order statistics) associated with sampling from corresponding distributions can be found easily and quickly making tables, simulation and special computer programs unnecessary.

The analytical methodology described in this paper is illustrated for the two-parameter Weibull and extreme-value distributions. Applications to other log-location-scale distributions could follow directly.

Finally, we give the three illustrative numerical examples, where the exact statistical tolerance limits with expected $(1-\alpha)$ -confidence, obtained in this paper in terms of the two-parameter Weibull or extreme-value distribution, are compared with the known results that are reported in the literature and were obtained by using the following: 1) tables, 2) simulation, 3) Monte Carlo estimated percentiles, 4) special computer programs, 5) approximation, 6) transformation of the two-parameter Weibull distribution to the extreme-value distribution, etc.

Although the details of the problems addressed in the paper can vary significantly from one industry to another, the focus is always on making more accurate decisions, rather than manually using guesses and intuitions, but rather from a scientific point of view using models and technologies implemented with disciplined processes and systems.

The methodologies described here can be extended in several different directions to solve various problems arising in practice.

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