International Journal of Advancements in Electronics and Electrical Engineering – IJAEEE
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Volume 8 : Issue 1 - [ISSN : 2319-7498] - Publication Date: 27 Dec, 2019

Orthogonal Matching Pursuit with a Normalized Residual Based Stopping Criterion

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Abstract—**Orthogonal matching pursuit (OMP) is a commonly used algorithm in compressed sensing (CS) for estimating a sparse vector/signal $\mathbf{x} \in \mathbb{R}^n$ from linear measurements $\mathbf{y} \in \mathbb{R}^m$, where $m \ll n$. There are two generally stopping criteria adopted in the iterative OMP. One, assuming the number of nonzero entries of the sparse vector $\mathbf{x}$ is known, stop the algorithm after exactly K iterations. The other halt the pursuit if the strength of the residual is smaller than some threshold. These two criteria respectively rely on certain knowledge about the signal and the environment/noise. We propose a normalized residual strength based stopping criterion, which can be employed without the information mentioned above. Numerical results show that under some circumstances, the proposed criterion leads to a smaller normalized signal reconstruction error as compared to that achieved by OMP with exact K iterations and the conventional residual strength based stopping criterion.**

Keywords—orthogonal matching prusuit (OMP), compressed sensing (CS), stopping criterion.

I. Introduction

In compressed sensing (CS), we consider the following linear measurements model

$$\mathbf{y} = \Phi\mathbf{x}, \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the sparse signal, and $\Phi \in \mathbb{R}^{m \times n}$ is the sensing matrix with $m \ll n$. Denote the support of $\mathbf{x}$ by $\text{supp}(\mathbf{x})$. The number of measurements is selected such that the cardinality $K = |\text{supp}(\mathbf{x})|$ of supp($\mathbf{x}$) satisfies $K \ll m$. The recovery of $\mathbf{x}$ based on only $m \ll n$ measurements $\mathbf{y}$ is in general non-unique. However, if $\Phi \in \mathbb{R}^{m \times n}$ satisfies the restricted isometry property (RIP), the perfect recovery of y based on x can be guaranteed [1]. $\Phi \in \mathbb{R}^{m \times n}$ is said to satisfy RIP of order $2K$ if there exists a constant $0 < \delta_{2K} < 1$, we have

$$(1 - \delta_{2K})\|\mathbf{x}\|_2^2 \le \|\Phi\mathbf{x}\|_2^2 \le (1 + \delta_{2K})\|\mathbf{x}\|_2^2. \tag{1.2}$$

The smallest constant δ_{2K} such that the inequality (1.2) holds is called the restricted isometry constant (RIC). Thus given a measurement $\mathbf{y}$, there exists only one K-sparse vector $\mathbf{x}$ such that $\mathbf{y} = \Phi\mathbf{x}$. With the knowledge of sparsity inherent in the signal $\mathbf{x}$, one can thus uniquely find the correct sparse signal $\mathbf{x}$. To be more realistic, the measurement is considered with some contamination

$$\mathbf{y} = \Phi\mathbf{x} + \mathbf{w}, \tag{1.3}$$

where $\mathbf{w} \in \mathbb{R}^n$ represents the noise term. The orthogonal matching pursuit (OMP) [2]–[6] is a greedy algorithm, which is able to stably recover the sparse signal $\mathbf{x}$ from the set of incomplete linear measurements $\mathbf{y} \in \mathbb{R}^m$ in a noisy environment. In particular, the OMP estimates an element of $\text{supp}(\mathbf{x})$ in one iteration. The stopping criterion of OMP typically relies on either the knowledge of the support size K or the strength of the residual. If the knowledge of K is available, one can straightforwardly apply OMP with K iterations. In this case, we have satisfactory performance of the signal reconstruction via OMP especially when the signal to noise ratio (SNR) is high. However, such an oracle-like assumption is obviously too ideal. Therefore, one resorts to a residual based stopping criterion. More specifically, after the jth iteration, *i*) OMP have detected j elements of $\text{supp}(\mathbf{x})$, and as well have estimated the corresponding j-sparse vector $\hat{\mathbf{x}}_j \in \mathbb{R}^n$ which achieves $\min_{\mathbf{x}_j} \|\mathbf{y} - \Phi\mathbf{x}_j\|_2$, and thus *ii*) the residual reads $\mathbf{r}_j = \mathbf{y} - \Phi\hat{\mathbf{x}}_j$. In the literature [4]–[6], we generally employ $\|\mathbf{r}_j\|_2 < \sigma_1$ as the stopping criterion. That is, the OMP estimates one support element iteratively until the strength of the residual is sufficiently small as compared to the threshold σ_1. Indeed, the performance of the lies largely on the value of σ_1. To determine a proper σ_1, one natural way is to first obtain the characteristic of the noise term $\mathbf{w}$ such as

$$\|\mathbf{w}\|_2 < \varepsilon. \tag{1.4}$$

Then we can set $\sigma_1 = \varepsilon$ [4]–[6]. Nevertheless, in reality it is not guaranteed that the noise strength is always bounded. Instead, the noise term is generally assumed to has a Gaussian distribution; thus, with an arbitrary but fixed ε, the noise strength is bounded by ε (i.e., $\|\mathbf{w}\|_2 < \varepsilon$) with an associated probability $P_\varepsilon \triangleq \Pr(\|\mathbf{w}\|_2 < \varepsilon)$. If σ_1 is

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This work is sponsored by the Ministry of Science and Technology, Taiwan, under grants MOST 107-2218-E-155-012-MY2 .

Table I. The OMP Algorithm. In Step 3.4, the entries of $\mathbf{q}_{\Omega^j} \in \mathbb{R}^j$ are those of the updated $\mathbf{q}$ corresponding to the index set Ω^j.

1. Input: $\mathbf{y}$, $\mathbf{\Phi}$
2. Initialize: $\mathbf{r}^0 = \mathbf{y}$, $\Omega^0 = [\ ]$, and $j = 0$
3. While stopping criterion is not satisfied 3.0 $j = j + 1$ 3.1 $p^j = \arg\max_i \left\|\langle \mathbf{\Phi}\mathbf{e}_i, \mathbf{r}^{j-1}\rangle\right\|$ 3.2 $\Omega_j = [\Omega_{j-1}\ p^j]$ 3.3 $\mathbf{q} = [0\ 0\ \cdots\ 0]^T \in \mathbb{R}^N$ **3.4** $\mathbf{q}_{\Omega_j} = \arg\min_{\mathbf{b}} \left\|\mathbf{\Phi}_{\Omega_j}\mathbf{b} - \mathbf{y}\right\|_2 = (\mathbf{\Phi}_{\Omega_j}^T\mathbf{\Phi}_{\Omega_j})^{-1}\mathbf{\Phi}_{\Omega_j}^T\mathbf{y}$ **3.5** $\mathbf{r}^j := \mathbf{y} - \mathbf{\Phi}_{\Omega^j}\mathbf{q}_{\Omega^j}$ end while
4. output: $\hat{\mathbf{x}} = \mathbf{q}$

chosen to be relatively small, P_{σ_1} is less than one. Then, the threshold σ_1 is improper when $\|\mathbf{w}\|_2 \geq \sigma_1$ (with probability $1 - P_{\sigma_1}$). On the other hand, if σ_1 is large enough, P_{σ_1} is near one. It in turn implies that $\|\mathbf{w}\|_2$ is generally smaller than σ_1, leading to an early halt due to the stopping criterion $\|\mathbf{w}\|_2 < \sigma_1$. As a result, determining a proper threshold σ_1 is a crucial, and can be a daunting, task in OMP. In this paper, we propose a stopping criterion based on the normalized strength of the residual. Namely, $\|\mathbf{r}_j\|_2 / \|\mathbf{y}\|_2 < \sigma_2$. Notably, we do not need the statistical property of $\mathbf{w}$. Let $\hat{\mathbf{x}}$ denotes the reconstructed sparse signal based on the OMP. It is shown from our numerical simulations that when $\sigma_2 = 0.1$, the normalized signal reconstruction error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 / \|\mathbf{x}\|_2$ converges to the one achieved by applying the OMP with exactly K iterations as the signal to noise ratio (SNR) increases to infinity either when $\mathbf{\Phi} \in \mathbb{R}^{154 \times 512}$ or when $\mathbf{\Phi} \in \mathbb{R}^{70 \times 512}$. The numerical simulations also show that the proposed stopping criterion results in smaller normalized signal reconstruction error as compared to the conventional residual based stopping criterion. When $\sigma_2 = 0.3$, the performance using the OMP with the proposed stopping criterion outperforms the OMP with K iterations when SNR is low in the sense that a smaller normalized signal reconstruction error is induced.

II. The Proposed Stopping Criterion and Discussions

The OMP algorithm is presented in Table 1. The generally employed stopping criteria are shown below:

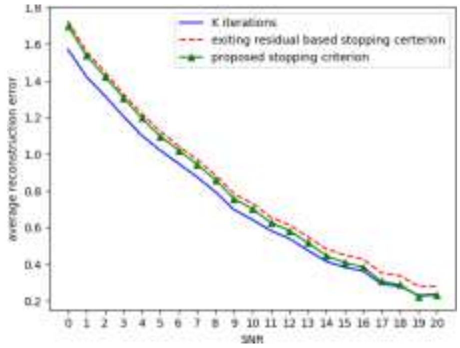
1) $j \geq K$.

2) $\|\mathbf{r}_j\|_2 < \sigma_1$.

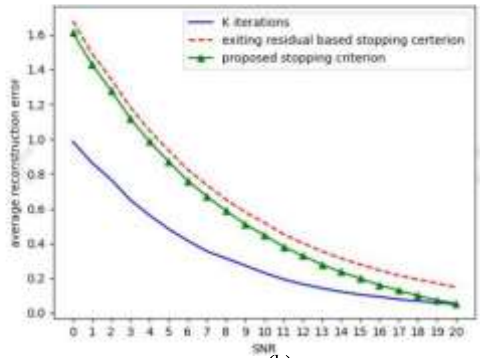
We note that the first condition relies on the knowledge of the sparsity of $\mathbf{x}$, which is quite an ideal information of the signal. Namely, criterion i) can be only applied under a rather strong assumption with rich information. Thus it can be regarded as the benchmark of the performance using the OMP. On the other hand, the determination of the threshold σ_1 in the second condition is the core to the signal reconstruction performance. An appropriate threshold can be selected on the basis of the statistical property of the noise term $\mathbf{w}$ [4]–[6]. In this paper, we propose to employ the following criterion instead:

$$3)\ \frac{\|\mathbf{r}^j\|_2}{\|\mathbf{y}\|_2} < \sigma_2 .$$

We can note that neither the knowledge of the sparsity K of the signal $\mathbf{x}$ nor the information about the noise is needed in the proposed criterion 3). According to criterion 3), we kind of compare the strengths of $\mathbf{r}_j$ and $\mathbf{y}$ without considering the noise term $\mathbf{w}$. In particular, if the residual strength $\|\mathbf{r}_j\|$ is smaller than the measurement strength $\|\mathbf{y}\|$ times σ_2, we stop the OMP and output the estimated sparse signal $\hat{\mathbf{x}}$. The threshold σ_2 determines "how much smaller" is required in the stopping criterion. We have some numerical simulations showing that the proposed stopping criterion outperforms than criterion 1) an criterion 2) in some circumstances. The ambient dimension is set to be n=512. The sensing matrix $\mathbf{\Phi}$ is randomly generated with every independent entry drawn from a common Gaussian distribution. From Fig. 1-(a), and Fig. 1-(b), we can see that if $\sigma_2 = 0.1$, the average normalized signal reconstruction error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 / \|\mathbf{x}\|_2$ induced by the proposed criterion converges to the one corresponding to the criterion 1) as the signal to noise ratio (SNR), defined as $20\log\frac{\|\mathbf{y}\|_2}{\|\mathbf{w}\|_2}$, increases to 20dB when the measurement dimension is either m=70 or m=154. From the basics of CS, it is known that there is a tradeoff between the reduction of the measurements size and the signal reconstruction performance. Thus we can see that when the measurements size is m=70, the resultant normalized signal reconstruction error with respect to any one stopping criterion is no smaller than that achieved by the same criterion when the measurements size is m=154. If $\sigma_2 = 0.3$, the simulation results (see Fig. 2-(a) and Fig. 2-(b)) show that the proposed criterion yields smaller average signal reconstruction error than the benchmark reconstruction error with respect to the OMP with K iterations (criterion 1)) when SNR is less than 14dB. This is not surprising because when SNR is low, the measurements are severely contaminated by the noise, and thus the rather weak components of the signal is almost irrecoverable. The proposed criterion suggests us that when

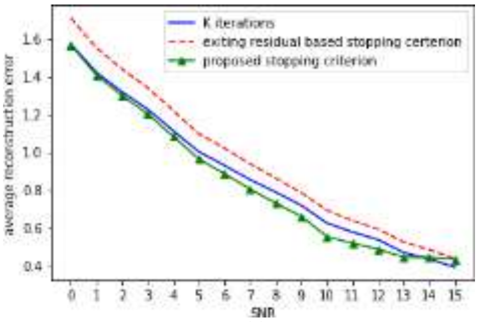


(a)

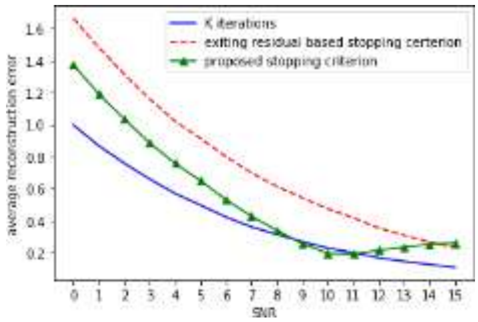


(b)

Fig. 1. The normalized reconstruction error of the OMP with respect to stopping criterions i) $j \geq K$, ii) $\| \mathbf{r}^j \|_2 < \sigma_1$, and iii) $\| \mathbf{r}^j \|_2 / \| \mathbf{y} \|_2 < \sigma_2$ with $\sigma_2 = 0.1$, under different SNR. (a) The sensing matrix is of dimension $\Phi \in \mathbb{R}^{70 \times 512}$. (b) The sensing matrix is of dimension $\Phi \in \mathbb{R}^{154 \times 512}$.



(a)



(b)

Fig. 2. The normalized reconstruction error of the OMP with respect to stopping criterions i) $j \geq K$, ii) $\| \mathbf{r}^j \|_2 < \sigma_1$, and iii) $\| \mathbf{r}^j \|_2 / \| \mathbf{y} \|_2 < \sigma_2$ with $\sigma_2 = 0.3$, under different SNR. (a) The sensing matrix is of dimension $\Phi \in \mathbb{R}^{70 \times 512}$. (b) The sensing matrix is of dimension $\Phi \in \mathbb{R}^{154 \times 512}$.

the normalized residual strength satisfies $\| \mathbf{r}_j \|_2 < \sigma_2 \| \mathbf{y} \|_2 = 0.3 \| \mathbf{y} \|_2$, we should stop estimating the remaining small entries of the signal. Thus, the resultant estimated signal $\hat{\mathbf{x}}$ is closer to $\mathbf{x}$. As the SNR increases to more than 14dB, the average signal reconstruction error with respect to the proposed criterion increases. This is mainly because when SNR is large, the noise vanishes, thus the remaining residual in every iteration is largely composed of the signal itself; as a result, one can conduct the OMP until the residual strength is very small since the weak nonzero entries of the signal can still be recovered. However, the OMP is stopped if $\| \mathbf{r}_j \|_2 < 0.3 \| \mathbf{y} \|_2$, which is fulfilled when the residual strength is not yet sufficiently small. In [7], a so-called orthogonal matching pursuit with thresholding (OMPT) is proposed in consideration with a noiseless environment. In particular, the index p^j selected in each iteration merely satisfies $\| \mathbf{r}^{p^j} \|_2 / \| \mathbf{y} \|_2 < \sigma_2$. That is, p^j does not yield the maximal $\left| \left\langle \Phi \mathbf{e}_i , \mathbf{r}^{j-1} \right\rangle \right|$ over all i, resulting in rather inaccurate detection of the support element in the noisy environment. Also, the suggested threshold σ_2 by [7] is very small (much less than 0.1). Thus the OMPT is not an ideal option when considering a noisy case.

References

[1] R. G. Baraniuk, "Compressive sensing," *IEEE Signal Processing Magazine*, vol. 24, pp. 118–124, July 2007.

[2] M. A. Davenport and M. B. Wakin, "Analysis of orthogonal matching pursuit using the restricted isometry property," *IEEE Trans. on Inf. Theory*, vol. 56, pp. 4395–4401, Sept. 2010.

[3] Q. Mo and Y. Shen, "A remark on the restricted isometry property in orthogonal matching pursuit," *IEEE Trans. on Inf. Theory*, vol. 58, pp. 3654–3656, June 2012.

[4] Ling-Hua Chang and Jwo-Yuh Wu, "An improved RIP-based performance guarantee for sparse signal recovery via orthogonal matching pursuit," *IEEE Trans. on Inf. Theory*, vol. 60, no. 9, pp. 5702-5715, Sept. 2014.

[5] R. Wu, W. Huang, and D.-R. Chen, "The exact support recovery of sparse signals with noise via orthogonal matching pursuit," *IEEE Signal Process. Lett.*, vol. 20, no. 4, pp. 403–406, Apr. 2013.

[6] T. T. Cai and L. Wang, "Orthogonal matching pursuit for sparse signal recovery with noise," *IEEE Trans. Inf. Theory*, vol. 57, no. 7, pp. 4680–4688, Jul. 2011.

[7] M. Yang, F. D. Hoog, "Orthogonal Matching Pursuit With Thresholding and its Application in Compressive Sensing," *IEEE Signal Process. Lett.*, vol. 63, no. 20, pp. 5470–5486, July. 2015.

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