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NUMERICAL SOLUTION OF NON-LINEAR FRACTIONAL BURGERS' EQUATION USING SINC–MUNTZ COLLOCATION METHOD

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Abstract. In this paper a new numerical method is presented for solving non-linear fractional Burgers' equation(FBE). The technique is based on the collocation method where the fractional Muntz-Legendre functions in time and the Sinc functions in space are utilized, respectively. By using these functions, we approximate the unknown functions. The proposed approximation together with collocation method reduce the solution of the FBE to the solution of a system of nonlinear algebraic equations. Finally, some numerical examples show the validity and accuracy of the present method.

1. Introduction

Fractional models are widely used in many physical models and engineering research. For this reason, from many years ago researchers have been interested in solving these types of equations [1, 2, 3]. Since the fractional equations contain fractional derivatives, they are unable to get the exact solution in many cases. We must resort to numerical method. Recently, several numerical techniques have been proposed by researchers for solving the fractional ordinary differential equations (FODEs) and the fractional partial differential equations (FPDEs). For example, Kazem and Abbasbandy [4] used fractional-order Legendre functions for solving FODEs. Esmaeili, Shamsi, and Luchko [5] applied a collocation method bases on Muntz polynomials for solving FODEs. Chen, Sun, and Liu [6] used generalized fractional-order Legendre function for solving FPDEs. The other numerical method can be found in [7, 8, 9, 10].

In this paper, we apply a numerical method for solving non-linear fractional Burgers' equation (FBE) as the following form:

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Key words and phrases. Sinc functions, fractional Muntz-Legendre polynomials, fractional Burgers' equation (FBE), collocation method, Caputo fractional derivative.

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$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \epsilon\frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t), \qquad (x,t)\in\Omega \tag{1}$$

with initial condition

$$u(x,0) = f_0(x), \tag{2}$$

and boundaryconditions

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t),$$

where $\Omega = (0,1)\times(0,1)$ and $0 < \alpha \le 1$ is the order of the fractional derivatives in the Caputo sense, and the continuous functions g and g_0 are known and the function $u(x,t)$ is unknown.

The aim of this paper is to apply the Sinc functions and Muntz–Legendre polynomials to achieve the numerical solution of problem (1).

This article is organized as follows: Review of Caputo fractional derivative and Review of fractional Muntz–Legendre polynomials is presented in Section **2.** In Section 3, we recall notations of the Sinc functions and their properties. In Sections 4 and 5, we discuss the convergence analysis and the approximate solution of the FBE using a collocation method based on Sinc functions and Muntz–Legendre polynomials. In Section 6, we present some examples of FBE to show efficiency and accuracy of the proposed method. Finally a conclusion is expressed in Section 7.

2. Preliminaries and notation

In this section, we give the definition and some properties of Caputo fractional derivative and fractional-order Muntz–Legendre polynomials.

2.1. Review of Caputo fractional derivative

Deftnition 1. The fractional derivative of $y(t)$ in the Caputo sense is defined as

$$D^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau)\,d\tau$$

for $m-1 < \alpha < m, m \in \mathbb{N}$, and $t > 0$.

Deftnition 2. Let $\alpha > 0$. The Riemann–Liouville fractional integral operator J_t^α, defined on $L_1[a,b]$ by

$$J_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} y(\tau)\,d\tau.$$

Some properties of the Riemann–Liouville fractional integral operator J_t^α and the Caputo fractional derivative operator D_t^α, which will be used later, are as follows;

1) $D_t^\alpha C = 0$, where C is a constant.

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2)

(3) $$D^{\alpha} t^{v} = \begin{cases} \frac{\Gamma(v+1)}{\Gamma(v+1-\alpha)} t^{v-\alpha}, & v \in \mathbb{N}, v \geq \lceil \alpha \rceil, \text{ or } v \notin \mathbb{N}, v > \alpha, \\ 0, & v \in \mathbb{N}_0, v < \lceil \alpha \rceil \end{cases}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ is the largest integer less than or equal to α. Also $N_0 = \{0, 1, \ldots\}$

3) Caputo fractional derivative is a linear operation,

$$D^{\alpha}\left(\sum_{i=1}^{n} a_i y_i(t)\right) = \sum_{i=1}^{n} a_i D^{\alpha} y_i(t)$$

4)

$$J_t^{\alpha}(J_t^{\beta} y(t)) = J_t^{\beta}(J_t^{\alpha} y(t)) = J_t^{\alpha+\beta} y(t), \quad \alpha, \beta > 0.$$

5)

$$J_t^{\alpha} t^{v} = \frac{\Gamma(v+1)}{\Gamma(\alpha+v+1)} t^{\alpha+v}.$$

6)

$$D^{\alpha}(J_t^{\alpha} y(t)) = y(t).$$

7)

(4) $$J_t^{\alpha}(D^{\alpha} y(t)) = y(t) - \sum_{i=0}^{n-1} y^{(i)}(0)\frac{t^{i}}{i!}, \quad n-1 < \alpha \leq n,\ t > 0.$$

For more details about the properties of Caputo fractional derivative operator and Riemann–Liouville fractional integral operator see [2].

2.2. Review of fractional-order Muntz polynomials

Deftnition 3. (see [9]) The fractional-order Muntz–Legendre polynomials on the interval $[0, T]$ are represented by the formula

(5) $$L_n(t;\alpha) = \sum_{k=0}^{n} C_{n,k}\left(\frac{t}{T}\right)^{k\alpha},$$

where

$$C_{n,k} = \frac{(-1)^{n-k}}{\alpha^n k!(n-k)!}\prod_{v=0}^{n-1}\left((k+v)\alpha+1\right).$$

The function $L_k(t;\alpha)$, $k = 0, 1, \ldots, n$ forms an orthogonal basis for $M_{n,\alpha} = \mathrm{Span}\{1, t^{\alpha}, \ldots, t^{n\alpha}\}$, $t \in [0, T]$. Also is satisfies

$$L_0(t;\alpha) = 1,$$

$$L_1(t;\alpha) = \left(\frac{1}{\alpha}+1\right)\left(\frac{t}{T}\right)^{\alpha} - \frac{1}{\alpha},$$

$$b_{1,n} L_{n+1}(t;\alpha) = b_{2,n}(t) L_n(t;\alpha) - b_{3,n} L_{n-1}(t;\alpha),$$

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where

$$b_{1,n}=a_{1,n}^{0,\frac{1}{\alpha}},\ b_{2,n}(t)=a_{2,n}^{0,\frac{1}{\alpha}-1}\left(2\left(\frac{t}{T}\right)^{\alpha}-1\right),\ b_{3,n}=a_{3,n}^{0,\frac{1}{\alpha}-1},$$

$$a_{1,n}^{\alpha,\beta}=2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta),$$

$$a_{2,n}^{\alpha,\beta}(x)=(2n+\alpha+\beta+1)[(2n+\alpha+\beta)(2n+\alpha+\beta+2)x+\alpha^2-\beta^2],\ a_{3,n}^{\alpha,\beta}=2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2).$$

Theorem 2.1. *Let* $L_n(t;\alpha)$ *be a fractional-order Muntz–Legendre polynomials, then we have the following Caputo fractional derivative of the functions* $L_n(t;\alpha)$*:*

$$D^{\alpha}L_n(t;\alpha)=\sum_{k=1}^{n}D_{n,k}\left(\frac{t}{T}\right)^{(k-1)\alpha}, \tag{6}$$

where

$$D_{n,k}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+k\alpha-\alpha)T^{\alpha}}C_{n,k},$$

and $C_{n,k}$ *is defined in* $L_n(t;\alpha)$.

Proof. It is the result of the equations (3) and (5). □

Theorem 2.2. *Let* $\alpha>0$ *be a real number and* $t\in[0,1]$*. Then*

$$L_n(t;\alpha)=P_n^{(0,\frac{1}{\alpha}-1)}(2t^{\alpha}-1).$$

where $P_n^{(\alpha,\beta)}$ *are the Jacobi polynomials with parameters* $\alpha,\beta>-1$, [14, 15].

Proof. (see [5]).

3. Sinc function and its properties

In this section, we recall notation and properties of the Sinc function, and derive useful formulas that will be used in this paper. The Sinc function is defined on R as [11]

$$\mathrm{Sinc}(x)=\begin{cases}\frac{\sin(\pi x)}{\pi x}, & x\neq 0,\\ 1, & x=0.\end{cases}$$

Let $g(x)$ be a function defined on R, and let $h>0$ be a step size. Consider the Whittaker cardinal function of g is defined by the series

$$C(g,h)(x)=\sum_{k=-\infty}^{\infty}g(kh)\mathrm{Sinc}\left(\frac{x-kh}{h}\right).$$

This series converge (see [11]), and the kth Sinc function is defined on R as

$$S(k,h)(x)=\mathrm{Sinc}\left(\frac{x-kh}{h}\right).$$

Now, for positive integer N, the function g can be approximated by truncating as follows:

$$C_N(g,h)(x) = \sum_{k=-N} g(kh)\mathrm{Sinc}\left(\frac{x-kh}{h}\right).$$

The properties of the Whittaker cardinal expansion have been extensively studied in [11] . These properties are derived in the infinite strip D_S-plane of the complex ω-plane, where, for $d > 0$,

$$D_S = \{w = u + iv : |v| < d \leq \pi/2\}.$$

To construct approximations on the interval [a,b], which is used in this paper, the eye-shaped domain in the z–plane (see [11]),

$$D_E = \{z = u + iv : |\arg(\frac{x-a}{b-x})| < d \leq \pi/2\},$$

is mapped conformally onto the infinite strip D_S via:

$$\omega = \psi(z) = \ln\left(\frac{x-a}{b-x}\right)$$

The basic functions on [a, b] are taken to be translated Sinc functions

$$S_k(x) \equiv S(k,h) \circ \psi(x) = \mathrm{Sinc}\left(\frac{\psi(x)-kh}{h}\right), \tag{7}$$

where $S(k, h)\circ\psi(x)$ is defined by $S(k, h)(\psi(x))$. The inverse map of $\omega = \psi(z)$ is

$$z = \psi^{-1}(\omega) = \frac{a + be^{\omega}}{1 + e^{\omega}}.$$

Thus we may define the inverse images of the real line and of the evenly spaced nodes

$$x_k = \psi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm1, \pm2, \ldots.$$

Deftnition 4. (see [12]) Let $B(D_E)$ be the class of functions g that are analytic in D_E, and they satisfy

$$\int_{\psi^{-1}(x+L)} |g(z)|dz \to 0, x \to \pm\infty$$

where

$$L = \{iy : |y| < d \leq \pi/2\},$$

and those on the boundary of D_E satisfy

$$\int_{\partial D_E} |g(z)|dz < \infty.$$

4. Convergence analysis

The following expressions show that the Sinc interpolation on $B(D_E)$ converges exponentially.

Theorem 4.1. *(see [11, 12]) Assume that* $g\psi' \in B(D_E)$*; then, for all* x *in* $[a, b]$,

$$|g(x) - \sum_{k=-\infty}^{\infty} g(kh)S(k, h) \circ \psi(x)| \leq \frac{2N(g\psi')}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|g(x)| \leq Ce^{\gamma|\psi(x)|}, x \in \Gamma$ *for some positive constants* C *and* γ, *and the selection* $h = \sqrt{\pi d/\gamma N} \leq 2\pi d/\ln(2)$, *then*

$$\left|\frac{d^n g(x)}{dx^n} - \sum_{k=-N}^{N} g(kh) \frac{d^n}{dx^n} S(k, h) \circ \psi(x)\right| \leq kN^{(n+1)/2} e^{-\sqrt{\pi d \gamma N}}$$

for all $n = 0, 1, 2, \ldots, m$.

Also, the nth derivative of the function g at some points x_k can be approximated (see [13]), as follows:

$$\delta^{(0)}_{k,j} = [S(k, h) \circ \psi(x)]|_{x=x_j} = \delta_{k,j},$$

where

$$\delta_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

It has been shown that

$$\delta^{(1)}_{k,j} = \frac{d}{d\psi}[S(k, h) \circ \psi(x)]|_{x=x_j} = \frac{1}{h} \begin{cases} 0, & j = k, \\ \frac{(-1)^{(j-k)}}{j-k}, & j \neq k, \end{cases}$$

and

$$\delta^{(2)}_{k,j} = \frac{d^2[S(k, h) \circ \psi(x)]|_{x=x_j}}{d\psi^2} = \frac{1}{h^2} \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{(j-k)}}{(j-k)^2}, & j \neq k. \end{cases}$$

So the approximate of a function $u(x)$ by Sinc expansion is

$$u_N(x, t) \simeq \sum_{i=-N}^{N} c_i S_i(x), \tag{8}$$

where $S_i(x)$ is defined in equations (7). Now, for orbitary $t_j \in (0, 1)$ and fix, we define $u(x_k) = u(x_k, t_j)$, then, To approximate the first and second derivatives at the Sinc nodes x_k, we have

(9)

$$\frac{\partial u_{N,n}(x_k, t_j)}{\partial x} = \frac{du(x_k)}{dx} = \frac{du_N(x_k)}{dx} + E_1 = \sum_{i=-N}^{N} c_i \left(\frac{d}{dx}[S_i(x)]\right)_{x=x_k} + E_1$$

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$$= \sum_{i=-N}^{N} c_i \left(\frac{d}{d\psi}[S(i,h)\circ\psi(x)]\frac{d\psi}{dx}\right)_{x=x_k} + E_1$$

$$= \sum_{i=-N}^{N} c_i\, \delta^{(1)}_{i,k}\frac{d\psi(x_k)}{dx} + E_1,$$

and,

(10)

$$\frac{\partial^2 u_{N,n}(x_k,t_j)}{\partial x^2} = \frac{d^2u(x_k)}{dx^2} = \frac{d^2 u_N(x_k)}{dx^2} + E_2 = \sum_{i=-N}^{N} c_i\left(\frac{d^2}{dx^2}[S_i(x)]\right)_{x=x_k} + E_2$$

$$= \sum_{i=-N}^{N} c_i\left(\frac{d^2\psi}{dx^2}\frac{d}{d\psi}[S(i,h)\circ\psi(x)] + \left(\frac{d\psi}{dx}\right)^2\frac{d^2}{d\psi^2}[S(i,h)\circ\psi(x)]\right)_{x=x_k} + E_2$$

$$= \sum_{i=-N}^{N} c_i\left(\frac{d^2\psi(x_k)}{dx^2}\delta^{(1)}_{i,k} + \left(\frac{d\psi(x_k)}{dx}\right)^2\delta^{(2)}_{i,k}\right) + E_2,$$

where

$$E_1 = O(N e^{-\sqrt{\pi d\gamma N}})$$

and,

$$E_2 = O(N^{\frac{3}{2}} e^{-\sqrt{\pi d\gamma N}}).$$

Moreover, the approximation of the first and second derivatives at the vector nodes x_k can be written as following form (see [16])

(11) $$u'(x_k,t_j) \simeq \left(\frac{-1}{h}I^{(1)}_m D(\psi) + I^{(0)}_m D\left(\frac{\psi}{\psi}\right)\right)u(x_k,t_j) = A\,u(x_k,t_j)$$

and,

(12) $$u''(x_k,t_j) \simeq \left(\frac{1}{h^2}I^{(2)}_m + hI^{(1)}_m D\left(\frac{\psi}{\psi}\right)\right)u(x_k,t_j) = B\,u(x_k,t_j)$$

where the $m \times m$, $(m = 2N + 1)$ Toeplitz matrices $I^{(q)}_m = [\delta^{(q)}_{j,k}]$, $q = 0, 1, 2$. i.e.,defined in [16]. By using the matrices in (11) and (12), with the notation $U = [u(x_i, t_j)]$, $tt = [g(x_i, t_j)]$ and $U^0 = [u(x_i, 0)]$, we can writte the equation (1) as the following system

$$D^{(\alpha)}_t U + U \circ AU - \epsilon BU = tt$$

where the symbol "$\circ$" means the Hadamard matrix multiplication. An alternative solution is to convert the FBE to an integral equation by operating with J^{α}_t on both sids of equation (1), and using properties (3)–(4), we have

$$u(x,t) = \sum_{i=0}^{m-1} u^{(i)}(x,0)\frac{t^i}{i!} + J^{\alpha}_t\left(g(x,t) - u(x,t)\frac{\partial u(x,t)}{\partial x} + \epsilon\frac{\partial^2 u(x,t)}{\partial x^2}\right).$$

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Since $0 < \alpha \leq 1$, we choose $m = 1$, then, we obtain the following system

$$\left({}_0D_t^{\alpha} \mathrm{U} = \mathrm{U} + \mathrm{J} \right)$$

Now, for the convergence proof of the solution $\mathrm{U} = \mathrm{U} \circ \mathrm{A}\mathrm{U}$ for this system can be done using fixed point Theory (see [16]–[18]). Thus the approximate solution will converge to the exact solution.

5. Approximate solution to the FBE by collocation method

In this section we approximate the solution of equation (1) by applying the Sinc function and fractional Muntz–Legendre polynomials which is discussed in the previous sections.

First, we approximate the unknown function $u(x, t)$ as follows:

$$u_{N,n}(x,t) \simeq \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} S_i(x) L_j(t;\alpha) \tag{13}$$

where $S_i(x)$ and $L_j(t;\alpha)$ are defined in equations (7) and (5), respectively. Moreover, let x_k be Sinc collocation points. Then we approximate the differential $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial^2 u(x,t)}{\partial x^2}$ and $\frac{\partial u(x,t)}{\partial t^{\alpha}}$ as follows:

$$\begin{aligned} \frac{\partial u_{N,n}(x_k,t)}{\partial x} &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} \left(\frac{d}{dx} [S_i(x)] \right)_{x=x_k} L_j(t;\alpha) \\ &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} \left(\frac{d}{d\psi} [S(i,h) \circ \psi(x)] \frac{d\psi}{dx} \right)_{x=x_k} L_j(t;\alpha) \\ &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij}\, \delta_{i,k}^{(1)} \frac{d\psi(x_k)}{dx} L_j(t;\alpha), \end{aligned} \tag{14}$$

(15)

$$\begin{aligned} \frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} \left(\frac{d^2}{dx^2} [S_i(x)] \right)_{x=x_k} L_j(t;\alpha) \\ &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} \left(\frac{d^2\psi}{dx^2} \frac{d}{d\psi} [S(i,h) \circ \psi(x)] + \left(\frac{d\psi}{dx}\right)^2 \frac{d^2}{d\psi^2} [S(i,h) \circ \psi(x)] \right)_{x=x_k} L_j(t;\alpha) \\ &= \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} \left(\frac{d^2\psi(x_k)}{dx^2} \delta_{i,k}^{(1)} + \left(\frac{d\psi(x_k)}{dx}\right)^2 \delta_{i,k}^{(2)} \right) L_j(t;\alpha), \end{aligned}$$

and

$$\frac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} = \sum_{i=-N}^{N} \sum_{j=0}^{n} a_{ij} S_i(x) D^{\alpha} L_j(t;\alpha), \tag{16}$$

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where $D^{\alpha}L_j(t;\alpha)$ is defined in Theorem 2.1.

Substi tuting equations (13)–(16) into the equation (1) and the initial condition (2), we get

(17) $$\frac{\partial^{\alpha} u_{N,n}(x_k,t)}{\partial t^{\alpha}} + u_{N,n}(x_k,t)\frac{\partial u_{N,n}(x_k,t)}{\partial x} - \epsilon\frac{\partial^2 u_{N,n}(x_k,t)}{\partial x^2} = g(x,t),$$

(18) $$u_{N,n}(x,0) = f_0(x).$$

Now, To find unknown coefficients a_{ij} in equations (17) and (18) , we use the collocation method with suitable collocation points (x_k, t_r), where $x_k = e^{kh}/(1+e^{kh})$, $h = \pi d/(\alpha N)$ and $d = \pi/2$ for $k = N, \ldots, N$, (see [11]) and t_r are Chebyshev–Gauss–Lobatto points with the following relation:

$$t_r = \frac{1}{2} - \frac{1}{2}\cos\frac{\pi r}{n}, \quad r = 1, \ldots, n.$$

Substituting these points into the equations (17) and (18), we get

$$\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij} S_i(x_k) D^{\alpha} L_j(t_r;\alpha) + \Big(\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij} S_i(x_k) L_j(t_r;\alpha)\Big)\Big(\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}\delta^{(1)}_{i,k}\frac{d\psi(x_k)}{dx} L_j(t_r;\alpha)\Big)$$

$$-\epsilon\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij}\Big(\frac{d^2\psi(x_k)}{dx^2}\delta^{(1)}_{i,k} + \Big(\frac{d\psi(x_k)}{dx}\Big)^2\delta^{(2)}_{i,k}\Big) L_j(t_r;\alpha) = g(x_k,t_r),$$

$$\sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij} S_i(x_k) L_j(0;\alpha) = g_0(x_k),$$

Now, we have a system of nonlinear algebraic equations with unknown coefficients a_{ij} by using the well known Newton's method; we can find the approximate solution

$$u_{N,n}(x,t) \simeq \sum_{i=-N}^{N}\sum_{j=0}^{n} a_{ij} S_i(x) L_j(t;\alpha).$$

6. Numerical illustration

In this section, we present some examples of linear and nonlinear of FBE to show the efficiency of the proposed method. The results will be compared with the exact solutions. The accuracy of present method is estimated by the absolute error $E_{N,n}$, which is given as follows:

$$E_{N,n} = |u(x_i,t_j) - u_{N,n}(x_i,t_j)|.$$

Example 1. *Consider the FBE.*

$$\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \epsilon\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad (x,t) \in (0,1)\times(0,1),$$

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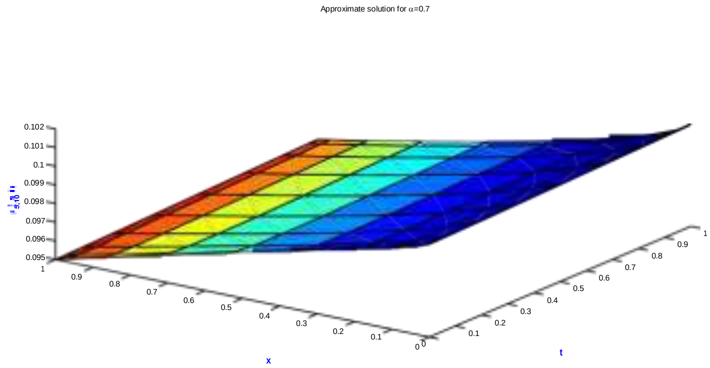

Figure 1. The approximate solution with $\alpha = 0.7$ for Example1.

$$u(x,0) = \frac{1}{10}\left(1 - \tanh\left(\frac{x}{20\epsilon}\right)\right),$$
$$u(0,t) = \frac{1}{10}\left(1 - \tanh\left(\frac{-t}{200\epsilon}\right)\right),$$
$$u(1,t) = \frac{1}{10}\left(1 - \tanh\left(\frac{1-0.1t}{20\epsilon}\right)\right).$$

Setting $\epsilon = 1$, the approximate solutions are shown in Figure 1 and Figure 2 for $\alpha = 0.7$ and $\alpha = 0.7$, respectively. The exact solution, for the special case $\alpha = 1$, is given by [16]

$$u(x,t) = \frac{1}{10}\left(1 - \tanh\left(\frac{x-0.1t}{20\epsilon}\right)\right).$$

Figure 3 shows comparison between the exact solution and the approximate solution for $\alpha = 1$. From Figure 3, we see that the obtained results are in good agreement with exact solution.

Example 2. *Consider the FBE.*

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t), \qquad 0 < \alpha \leq 1,$$
$$u(x,0) = x^2, \quad u(0,t) = t^2, \quad u(1,t) = 1 + t^2,$$

where

$$g(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2(x^3 + xt^2 - 1).$$

The exact solution is $u(x,t) = x^2 + t^2$.

For various values of N, n and α we obtain approximate solution of this equation. The absolute error is shown in Figures 4, 5 and 6. Also, Table 1 shows the maximum absolute error for the various values of N, n and α. We see that the absolute error converges to zero as $N, n \to \infty$.

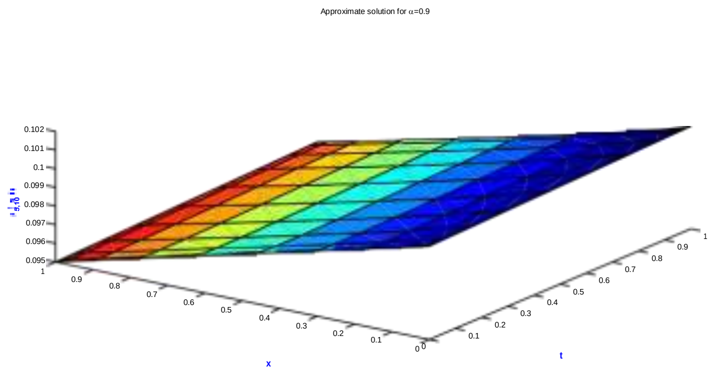

Figure 2. The approximate solution with α = 0.9 for Example1.

Table 1. The maximum absolute error for the various values of N, n, and α, for Example 2.

α	n = 6, N = 3	n = 8, N = 4	n = 10, N = 5
$\frac{3}{4}$	8.0677e−05	1.6679e−05	4.97788e−06
$\frac{1}{5}$	1.3901e−03	2.2598e−05	1.9007e−13
$\frac{3}{5}$	1.7979e−05	2.3834e−06	5.1442e−07

7. Conclusion

In this paper, we applied a basis of Sinc function and fractional Muntz–Legendre polynomials to obtain the numerical solution of nonlinear FBE. To get the unknown coefficients FMLPs, we use the collocation method. The results of the numerical examples show the efficiency and accuracy of the proposed method.

References

[1] Hilfer, R., Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, 2000.
[2] Podlubny, I., Fractional Differential Equations, Academic Press, Inc., San Diego, SA, 1999.
[3] Podlubny, I., Kacenak, M., MATLAB implementation of the Mittag-Leffler function, 2005-2009. http://www.mathworks.com.
[4] S. Kazem, S. Abbasbandy, S. Kumar, Fractional-order Legendre functions for solving fractional-order differential equations. Appl. Math. Model., 37 (7), 5498–5510 (2013).
[5] S. Esmaeili, M. Shamsi, Y. Luchko, Numerical solution of fractional differential equations with a collocation method based on Müntz polynomials. Comput. Math. Appl., 62(3), 918–929 (2011).
[6] Y. Chen, , Y. Sun, L. Liu, Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions. Appl. Math. Comput., 244, 847–858 (2014).

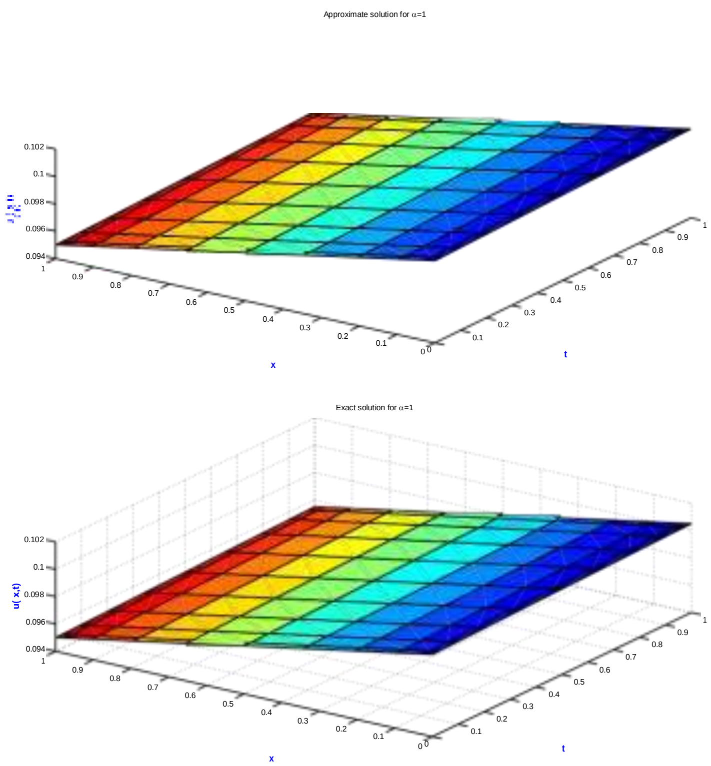

Figure 3. The approximate solution and the exact solution with α = 1 for Example1.

[7] Darani, A. M., M. Nasiri, A fractional type of the Chebyshev polynomials for approximation of solution of linear fractional differential equations. Computational Methods for Differential Equations (CMDE), 1(2), 96–107 (2013).

[8] N. Liu, E. B. Lin, Legendre wavelet method for numerical solutions of partial differential equations. Numer. Methods Partial Differential Equations, 26(1), 81–94 (2010).

[9] S. Abbasbandy, S. Kazem, M. S. Alhuthali, H. Alsulami, Application of the operational matrix of fractional-order Legendre functions for solving the time-fractional convection–diffusion equation. Appl. Math. Comput., 266, 31–40 (2015).

[10] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations. Comput. math. Appl., 59(3), 1326-.1336 (2010).

[11] F. Stenger , Handbook of Sinc Numerical Methods, Chapman & Hall/CRC Numerical Analysis and Scientific Computing. CRC Press, Boca Raton, FL,2011.

[12] F. Stenger , Numerical Methods Based on Sinc and Analytic Functions, Springer, Series in Computational Mathematics, 20. Springer-Verlag, New York 1993, NY, USA, 1993.

[13] M. El-Gamel , S.H. Behiry , H. Hashish , Numerical method for the solution of special nonlinear fourth-order boundary value problems, Appl. Math. Comput. 145 (2-3), 717–-734 (2003).

[14] W. Gautschi, Orthogonal Polynomials: Computation and Approximation, Oxford University Press, New York, 2004.
[15] G. Szegő, Orthogonal Polynomials, 4th ed., AMS Colloq. Publ, 1975.
[16] K. AL-KHaled, Sinc–Legendre collocation method for the non-linear Burgers' fractional equation, Annals of the university of Craiova, MAthematics and Computer Science Serics, 41(2), 234–250 (2014).
[17] M. M. Rodrigues, Study of solutions of a nonlinear fractional partial differential equations, Proceedings of the world congress on Engineering, WCE, 1, 2011.
[18] G. Amarand and D. Noureddine, Existence and uniqueness of solution to fractional Burgers' equation, Acta Univ. Apulensis, 21, 161–170 (2010).

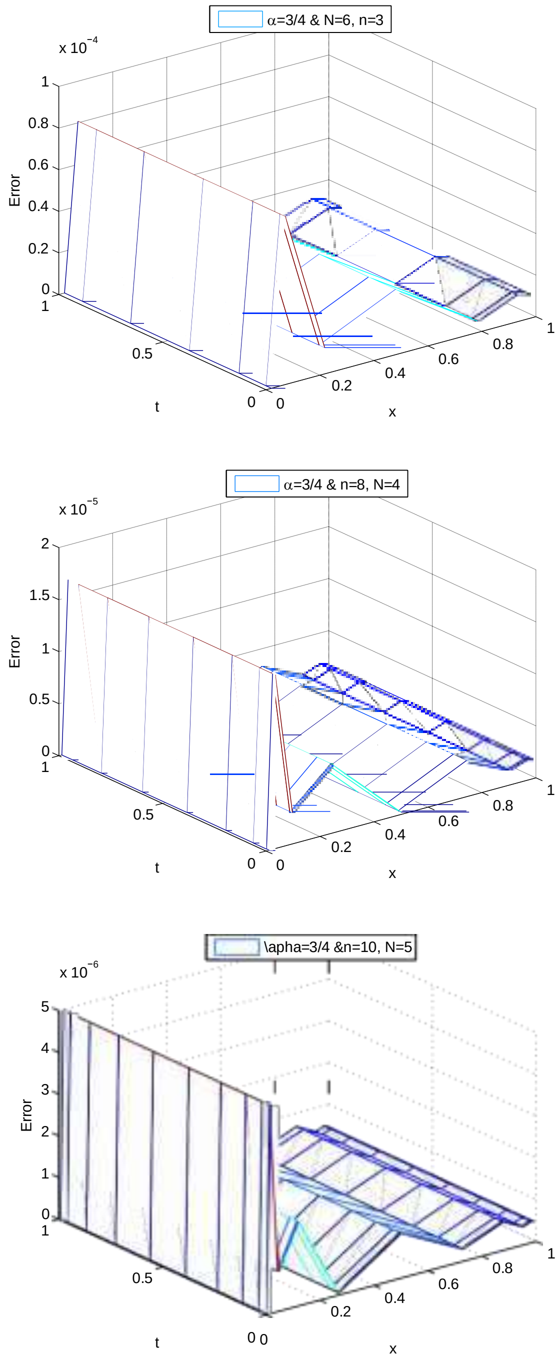

Figure 4. The absolute error function with $\alpha = \frac{3}{4}$ and various values of N, n for Example2.

NUMERICAL SOLUTION OF NON-LINEAR F-BE USING S–MC METHOD

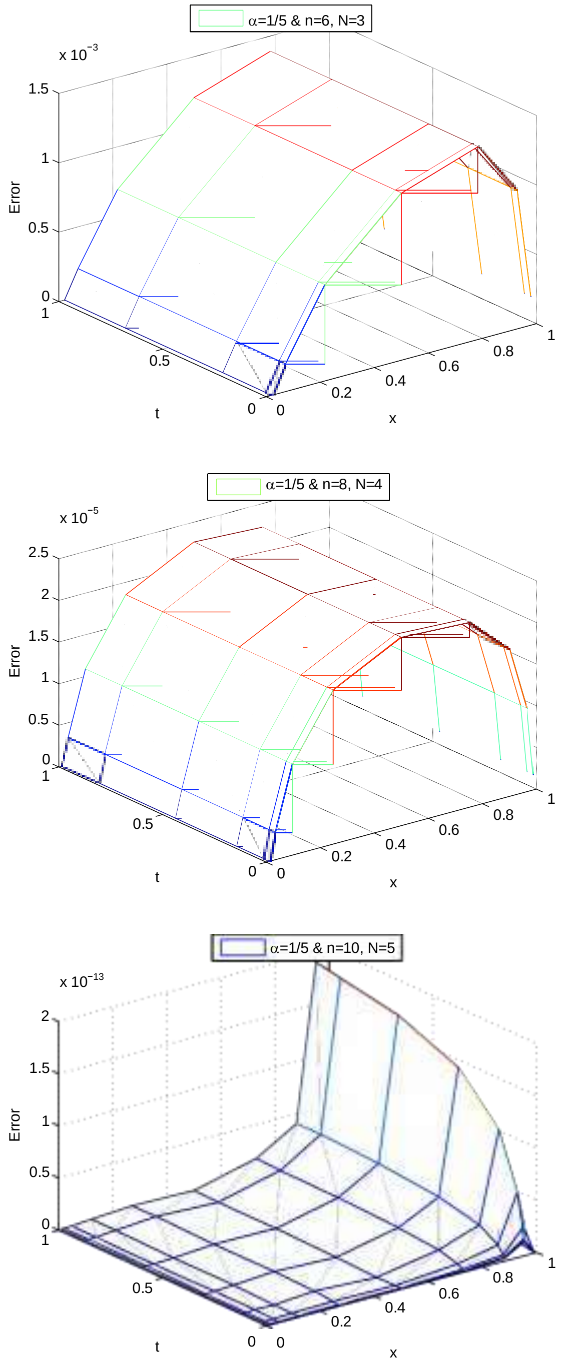

Figure 5. The absolute error function with $\alpha = \frac{1}{5}$ and various values of N, n for Example2.

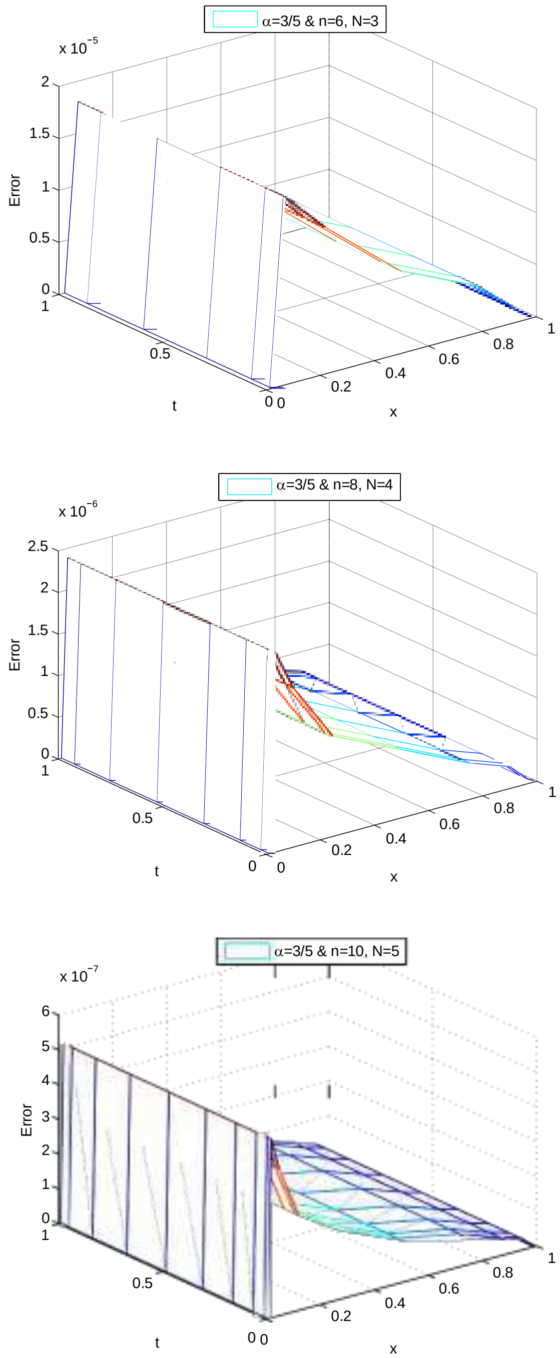

Figure 6. The absolute error function with $\alpha = \frac{3}{5}$ and various values of N, n for Example2.