*International Journal of Mathematical And Computational Science – IJMCS 2018*

*Volume 1 : Issue 1- [ISSN : 2475-2282] - Publication Date: 28 December, 2018*

# One step method for boundary control problem of unsteady Burgers equatıon

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***Abstract*— The paper deals with the all-at-once type solution of the boundary control problems for unsteady Burgers equation. The discretization of the problem leads to a system in saddle point form. Such problems need preconditioning when an iterative method is used. So that, related solution algorithms with appropriate preconditioners are described and analyzed.**

***Keywords*—optimal boundary control, Burgers equation, one-step method.**

## I. Introduction

In recent years, the so called all-at-once type methods were applied to PDE constrained OCP's [6, 10, 11]. Because all-at-once methods treat the control and state as independent optimization variables, the optimization problem is explicitly constrained; the state, control and adjoint state variables can be solved explicitly for all time-steps at once by solving a large system of linear equations. Analysis and numerical approximation of optimal control problems (OCP) for Burgers equation are important for the development of the numerical methods for the optimal control of more complicated models in fluid dynamics like Navier-Stokes equations [2, 3, 5]. The distributed and boundary optimal control problems for stationary and unsteady Burgers equation are solved using SQP (sequential quadratic programming) methods, primal-dual active set and semi-smooth Newton methods [1, 4, 7, 8, 9].

## II. Problem Formulation

We consider the unsteady Burgers equation with Robin boundary conditions

$$y_t + yy_x - \nu y_{xx} = f \text{ in } Q$$

with boundary conditions

$$\begin{aligned} y_t + yy_x - \nu y_{xx} &= f \text{ in } Q, \\ \nu y_x(\cdot,0) + \sigma_0 y(\cdot,0) &= u \text{ in } (0,T), \\ \nu y_x(\cdot,1) + \sigma_1 y(\cdot,1) &= v \text{ in } (0,T), \\ y(0,\cdot) &= y_0 \text{ in } \Omega, \end{aligned}$$

We consider the following optimal control problem for the unsteady Burgers equation with Robin boundary conditions

$$\min J(y,u,v) = \frac{1}{2}\int_Q (y-y_d)^2 dx + \frac{1}{2}\int_0^T \beta_u |u|^2 + \beta_v |v|^2 dt$$

subject to

$$\begin{array}{rcll} y_t + yy_x - \nu y_{xx} & = & f & \text{in } Q, \\ \nu y_x(\cdot,0) + \sigma_0 y(\cdot,0) & = & u & \text{in } (0,T), \\ \nu y_x(\cdot,1) + \sigma_1 y(\cdot,1) & = & v & \text{in } (0,T), \\ y(0,\cdot) & = & y_0 & \text{in } \Omega. \end{array}$$

## III. Semi implicit Scheme

Semi-implicit time approximation consists in evaluating the diffusive part at the time level i+1, whereas the remaining parts are considered at time level i. When this scheme is applied to a non-linear advection, it provides an efficient linearization. The main motivation for dealing with the diffusive part implicitly and the advection part explicitly for diffusion-convection equations is that the semi-implicit scheme is unconditionally stable and at each step a linear system with a symmetric matrix has to be solved. Because of these the semi-implicit scheme provides an effective linearization procedure for problems with non-linear advection terms like Burgers equation.

In order to get space discretization, we use standard Galerkin method with linear finite elements on the interval (0, 1) with n uniform subdivisions:

$$y_h(x,t) = \sum_{j=1}^{n-1} y_j(t)\phi_j(x).$$

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*Volume 1 : Issue 1- [ISSN : 2475-2282] - Publication Date: 28 December, 2018*

We obtain the semi-discrete system

$$M_h \frac{\vec{y}(t)}{dt} + \nu A_h \vec{y}(t) + q_h(\vec{y}(t)) = 0 \quad t \in (0, T)$$

where

$$q_h(\vec{y}(t)) := \frac{1}{6} \begin{pmatrix} y_1(t)y_2(t) + y_2^2(t) - u(t) - \sigma_1 y(1,t) \\ \vdots \\ -y_{i-1}(t)(y_{i-1}(t) + y_i(t)) + y_{i+1}(t)(y_i(t) + y_{i+1}(t)) \\ \vdots \\ -y_{n-2}^2(t) - y_{n-2}y_{n-1}(t) + v(t) - \sigma_2 y(nx+1,t) \end{pmatrix}$$

also $M_h$ and $A_h$ are the mass and stiffness matrices, respectively.

Semi-implicit time scheme is applied to semi-discrete problem to get

$$(M + \Delta t S)\mathbf{y}^{i+1} - M\mathbf{y}^i + \Delta t q(\mathbf{y}^i) = \Delta t M \mathbf{u}^{i+1},$$
$$\mathbf{y}(0) = \mathbf{y}_0 \quad \text{for } i = 0, ..., N.$$

## IV. Time stepping and discrete adjoint

Euler methods, Crank-Nicolson, Runga-Kutta methods, and various multistep methods are available as part of numerical tools of mathematical software. An efficient time integration scheme can be chosen for state and adjoint state. But in general the state and adjoint state are not adjoint anymore. In gradient based algorithms, such as CG-like methods, the solution of the discretized state equation is computed and the adjoint state is applied to calculate gradient. This may leads to additional errors. So that the discrete adjoint concept is introduced [5, 18]. We give the formulation for the backward Euler method.

The optimality system containing first order optimality conditions is obtained by introducing the extended Lagrangian containing the Lagrange multiplier *P*

$$L(Y, U, P) := \frac{\Delta t}{2}(Y - Y_d)^T \mathcal{M}_{1/2}(Y - Y_d) + \frac{\alpha \Delta t}{2} U^T \mathcal{M}_{1/2} U + P^T(-\mathcal{K}Y + \Delta t \mathcal{M} U + Q + d).$$

The optimality conditions are given as

$$\nabla_Y L(Y^*, U^*, P^*) = \Delta t \mathcal{M}_{1/2}(Y^* - Y_d) - \mathcal{K}^T P^* = 0,$$
$$\nabla_P L(Y^*, U^*, P^*) = -\mathcal{K}Y^* + \Delta t \mathcal{M} U^* + Q + d = 0,$$
$$\nabla_U L(Y^*, U^*, P^*) = \alpha \Delta t \mathcal{M}_{1/2} U^* + \Delta t \mathcal{M} P^* = 0.$$

Using the optimality conditions, the optimality system follows

$$\begin{pmatrix} \Delta t \mathcal{M} & 0 & 0 & -\mathcal{K}^T \\ 0 & \beta_v \Delta t \mathcal{L}_2 & 0 & \Delta t \mathcal{L}_2 \\ 0 & 0 & \beta_u \Delta t \mathcal{L}_1 & -\Delta t \mathcal{L}_1 \\ -\mathcal{K} & \Delta t \mathcal{L}_2 & -\Delta t \mathcal{L}_1 & 0 \end{pmatrix} \begin{pmatrix} Y \\ V \\ U \\ P \end{pmatrix} = \begin{pmatrix} \mathcal{M} Y_d \\ 0 \\ 0 \\ Q + d \end{pmatrix}$$

where

$$L_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \ddots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ & & 0 & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

With

$$\mathcal{L}_1 = \text{blockdiag}\{L_1, \dots, L_1\} \text{ and } \mathcal{L}_2 = \text{blockdiag}\{L_2, \dots, L_2\}$$

Also,

$$Q = (q_h(\mathbf{y}^1), ..., q_h(\mathbf{y}^N))$$

$$d = \begin{pmatrix} -M\mathbf{y}_0 + \Delta t q(\bar{\mathbf{y}}^0) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The first optimality condition gives the discrete adjoint equation

$$-\mathcal{K}^T P^* = \Delta t \mathcal{M}_{1/2}(Y^* - Y_d)$$

## V. Preconditioning

The system matrix arising from semi-implicit time approximation schemes for boundary control problem correspond to a saddle point system. Indeed, for the discrete problem we let

$$E := \begin{pmatrix} \mathcal{M} & 0 & 0 \\ 0 & \beta_v \Delta t \mathcal{L}_1 & 0 \\ 0 & 0 & \beta_v \Delta t \mathcal{L}_2 \end{pmatrix}$$

Also defining

$$x := \begin{pmatrix} Y \\ V \\ U \\ P \end{pmatrix}, \quad L := \begin{pmatrix} -\mathcal{K} & \Delta t \mathcal{L}_1 & \Delta t \mathcal{L}_2 \end{pmatrix}$$

and

$$b := \begin{pmatrix} \mathcal{M}_{1/2} Y_d \\ 0 \\ 0 \\ Q + d \end{pmatrix}$$

We define the following precondition

$$\mathcal{P} = \begin{pmatrix} \Delta t \mathcal{M} & 0 & 0 & 0 \\ 0 & \beta_v \Delta t \bar{\mathcal{L}}_2 & 0 & 0 \\ 0 & 0 & \beta_v \Delta t \bar{\mathcal{L}}_1 & 0 \\ 0 & 0 & 0 & S \end{pmatrix}$$

With

$$S^{-1} := \mathcal{K}^{-T} \mathcal{M} \mathcal{K}^{-1}.$$

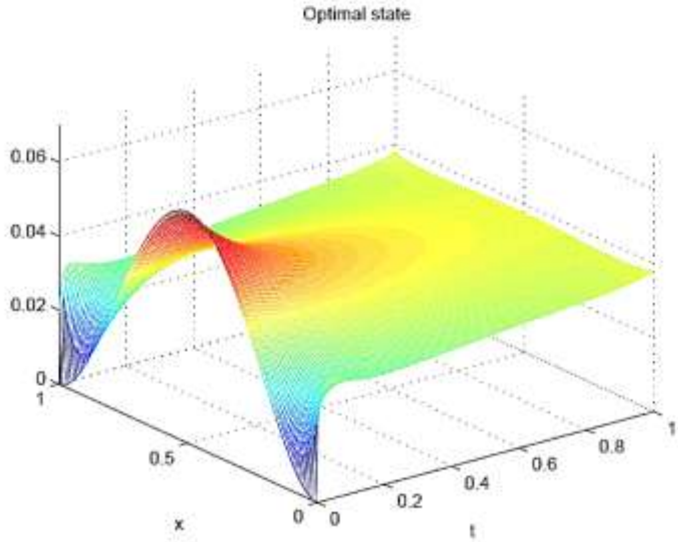
$$\bar{\mathcal{L}}_1 := \begin{pmatrix} \bar{L}_1 & & & & \\ \bar{L}_1 & \bar{L}_1 & & & \\ & \ddots & \ddots & & \\ & & \bar{L}_1 & \bar{L}_1 & \\ & & & \bar{L}_1 & \bar{L}_1 \end{pmatrix} \text{ and } \bar{\mathcal{L}}_2 := \begin{pmatrix} \bar{L}_2 & & & & \\ \bar{L}_2 & \bar{L}_2 & & & \\ & \ddots & \ddots & & \\ & & \bar{L}_2 & \bar{L}_2 & \\ & & & \bar{L}_2 & \bar{L}_2 \end{pmatrix}$$

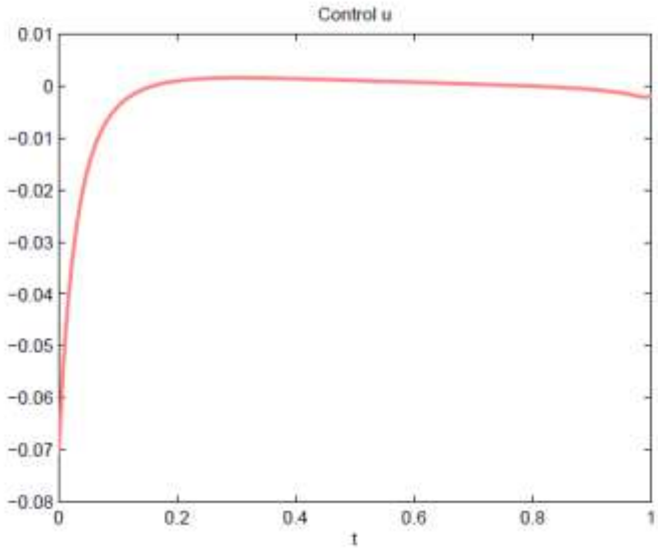
where

$$\bar{L}_1 := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \sigma & & \\ \vdots & & \ddots & \\ 0 & & & \sigma \end{pmatrix}, \quad \bar{L}_2 := \begin{pmatrix} \sigma & & \dots & 0 \\ \vdots & \ddots & & \\ & & \sigma & 0 \\ 0 & & 0 & 1 \end{pmatrix}.$$

## VI. Numerical Example

Let $Q = (0, 1) \times (0, 1)$. We consider a Neumann-type boundary control problem with $\beta_u = 0.01$, $\beta_v = 0.01$, $\sigma_0 = \sigma_1 = 0$. The viscosity parameter is $\nu = 0.1$. The initial condition is taken as $y_0 = x^2(1-x)^2$ and the desired state is $y_d = y_0$.



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