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The stochastic Heat Equation With Ito-Taylor Expansion

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Abstract— **In this paper, we analyze the numerical solution of the stochastic heat equation by using Ito-Taylor series expansion. We first use the finite difference method to discretize the space variable. Then, we apply Ito formula successively to get the numerical scheme.**

Keywords—The stochastic heat equation, finite difference, Ito-Taylor expansion.

I. Introduction

There has been a great interest in the numerical methods of SDEs in finance, medicine and the modern technologies [2, 3, 4, 6, 7]. As a result of the growth in science and the applications, SDEs are an emerging subject of interest. Stochastic Taylor expansion provides a source for the discrete-time approximation methods. One of the simplest ways to discretize the process is Euler method. The Milstein scheme [11], which has the order 1.0 of strong convergence, is stronger than Euler method. By adding more stochastic integrals using the stochastic Taylor expansion, more accurate schemes can be obtained [8].

Recently, there have been various studies of stochastic partial differential equations [1, 5, 9, 10]. Generally, we can not find the analytical solutions of SPDEs. So that numerical methods have been getting an important issue. In this work, we consider the stochastic heat equation. We apply the finite difference scheme to get a discretization in space. Then, the integral form of the eqution is considered with Ito formula. It is applied to the heat equation successively to get the numerical scheme.

II. Stochastic heat equation

We consider the one dimensional stochastic partial differential equation of the form

$$\begin{cases} du(t,x) = \left(\dfrac{\partial^2}{\partial x^2} u(t,x)\right) dt + dZ_t\,, & t \geq 0, \\ u(0,x) = u_0(x), & 0 \leq x \leq 1, \\ u(t,x) = u(t,1) = 0, & t \geq 0. \end{cases}$$

where Z_t denotes the one dimensioanl additive Brownian motion.

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III. Discretization

We apply the Ito-Taylor method to the time variable, we use a finite difference scheme to approximate the space variable.

For space variable, $0 \leq x_0 \leq x_1 \leq \cdots \leq x_j \leq \cdots \leq x_M = 1$ denotes the equispaced discretization of time space interval [0,1]. Then, by applying the second-order central difference scheme for the space variable, we get

$$du_t^j = \left(\frac{u_t^{j-1} - 2u_t^j + u_t^{j+1}}{h^2}\right) dt + dZ_t$$

Here, $h = x_{j+1} - x_j = \frac{1}{M}$ $\quad (j = 0,1,\ldots,M-1)$ and for $u_t^0 = u_t^M = 0$ we have

$$du_t^1 = \left(\frac{u_t^0 - 2u_t^1 + u_t^2}{h^2}\right) dt + dZ_t,$$

$$du_t^2 = \left(\frac{u_t^1 - 2u_t^2 + u_t^3}{h^2}\right) dt + dZ_{t,}$$

$$\vdots$$

$$du_t^j = \left(\frac{u_t^{j-1} - 2u_t^j + u_t^{j+1}}{h^2}\right) dt + dZ_{t,}$$

$$\vdots$$

$$du_t^M = \left(\frac{u_t^{M-1} - 2u_t^M + u_t^{M+1}}{h^2}\right) dt + dZ_t$$

In matrix notation, we have

$$d\mathbb{U}_t = \mathbb{A}_t \mathbb{U}_t dt + d\mathbb{Z}_t$$

$$\mathbb{A}_t = \begin{pmatrix} \frac{-2}{h^2} & \frac{1}{h^2} & 0 & \cdots & \cdots & \cdots \\ \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 & \cdots & \cdots \\ 0 & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \frac{1}{h^2} & \frac{-2}{h^2} \end{pmatrix}_{(M-1)\times(M-1)}$$

$\mathbb{U}_t = (u_t^1, u_t^2, \ldots, u_t^{M-1})^T$, and $d\mathbb{Z}_t = (dZ_t, \ldots, dZ_t)^T$

j^{th} component of the Euler scheme can be obtained as follows:

$$u_{t+1}^{j}=u_{t}^{j}+\left(\frac{u_{t}^{j-1}-2u_{t}^{j}+u_{t}^{j+1}}{h^{2}}\right)\Delta+\Delta Z$$
$$j\ =(1,2,\dots,M-1)$$

j^{th} component of theMilstein scheme is

$$u_{t+1}^{j}=u_{t}^{j}+\left(\frac{u_{t}^{j-1}-2u_{t}^{j}+u_{t}^{j+1}}{h^{2}}\right)\Delta+\Delta Z+\frac{1}{2}((\Delta Z^{j})^{2}-\Delta)$$
$$j=(1,2,\dots,M-1)$$

IV. Ito-Taylor Expansion

In this section, we will assume that the Brownian motions are independent. Now, consider

$$\mathrm{d}\mathbb{X}_{t}=\mathbb{A}\mathrm{d}t+\mathbb{H}\mathrm{d}\mathbb{Z}_{t}\qquad(1)$$

Eqn. (1) can be written in integral form as

$$\mathbb{X}_{t}=\mathbb{X}_{t_{0}}+\int_{t_{0}}^{t}\mathbb{A}(\mathrm{s},\mathbb{X}_{s})ds+\int_{t_{0}}^{t}\mathbb{H}(\mathrm{s},\mathbb{X}_{s})d\mathbb{Z}_{s}$$

The second integral is called *Ito stochastic integral*, which is defined by K. Ito in 1940. This integral can be approximated by stochastic Taylor method.

We consider now the k^{th} component of the system of stochastic differential equations in Eqn. 1

$$dX_{t}^{k}=a_{k}(t,\mathbb{X}_{t})dt+\sum_{j=1}^{n}h_{kj}(t,\mathbb{X}_{t})d\mathbb{Z}_{t}^{j}$$
$$(k=1,2,\dots,d)$$

Here, $a_{k}(t,\mathbb{X}_{t})=(t,X_{t}^{1},\dots,X_{t}^{d})$ ve $h_{kj}(t,\mathbb{X}_{t})=(t,X_{t}^{1},\dots,X_{t}^{d})$

And,

$$X_{t}^{k}=X_{t_{0}}^{k}+\int_{t_{0}}^{t}a_{k}(s,\mathbb{X}_{s}^{k})ds+\int_{t_{0}}^{t}h_{kj}(t,\mathbb{X}_{s}^{k})d\mathbb{Z}_{s}^{k}\qquad(2)$$

We will apply Ito-Lemma to Eqn. 2. Let

$g(t,\mathbb{X}_{t})=(g_{1}(t,\mathbb{X}_{t}),\dots,g_{p}(t,\mathbb{X}_{t}))^{T}$ and $Y_{t}=g(t,\mathbb{X}_{t})$

$$dY_{t}^{l}=\frac{\partial g^{l}}{\partial t}dt+\sum_{i=1}^{d}\frac{\partial g^{l}}{\partial x^{i}}dX_{t}^{i}+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^{2}g^{l}}{\partial x^{i}x^{j}}dX_{t}^{i}dX_{t}^{j}$$

where all derivatives of gℓ are to be evaluated in (t,Xt) and Brownian motions are uncorrelated and, in here, we note that ℓ and k represent the ℓth component of the function $g(t,\mathbb{X}_{t})$ and the kth component of the system of SDEs of Eqn. (1), respectively. Before obtaining the Taylor series expansions, we define the following operators:

$$\mathcal{L}^{0}:=\frac{\partial}{\partial t}+\sum_{i=1}^{d}a_{i}\frac{\partial}{\partial x^{i}}+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{p=1}^{n}h_{jp}h_{ip}\frac{\partial^{2}}{\partial x^{i}\partial x^{j}},$$

ve

$$\mathcal{L}^{j}=\sum_{p=1}^{d}h_{pj}\frac{\partial}{\partial x^{p}}\qquad(j=1,2,\dots,n)$$

We have

$$Y_{t}^{l}=Y_{t_{0}}^{l}+\int_{t_{0}}^{t}\mathcal{L}^{0}g^{l}ds+\sum_{j=1}^{n}\int_{t_{0}}^{t}\mathcal{L}^{j}\,g^{l}dZ_{s}^{j}$$

Let $g^{l}:=a_{l}(t,\mathbb{X}_{t})$ $(k=l)$ and $h^{l}:=h_{lj}(t,\mathbb{X}_{t})$. Ito Lemma gives

$$X_{t}^{k}=X_{t_{0}}^{k}+\int_{t_{0}}^{t}\left[a_{k}(t_{0},\mathbb{X}_{t_{0}})+\int_{t_{0}}^{s}\mathcal{L}^{0}a_{k}(\tau,\mathbb{X}_{\tau})d\tau+\sum_{j=1}^{n}\int_{t_{0}}^{s}\mathcal{L}^{j}a_{k}(\tau,\mathbb{X}_{\tau})dZ_{\tau}^{j}\right]ds+\sum_{j=1}^{n}\int_{t_{0}}^{t}\left[h_{kj}(t_{0},\mathbb{X}_{t_{0}})+\int_{t_{0}}^{s}\mathcal{L}^{0}h_{kj}(\tau,\mathbb{X}_{\tau})d\tau+\sum_{l=1}^{n}\int_{t_{0}}^{s}\mathcal{L}^{l}h_{kj}(\tau,\mathbb{X}_{\tau})dZ_{\tau}^{l}\right]dZ_{s}^{j}$$

We apply Ito lemma to the functions $\mathcal{L}^{0}a_{k}$, $\sum_{j=1}^{n}\mathcal{L}^{j}a_{k}$, $\mathcal{L}^{0}h_{kj}$, $\sum_{l=1}^{n}\mathcal{L}^{l}h_{kj}$, $\sum_{j,p=1}^{n}\mathcal{L}^{j}\mathcal{L}^{p}\,h_{kj}$, $\mathcal{L}^{0}\mathcal{L}^{0}a_{k}$, $\sum_{j=1}^{n}\mathcal{L}^{0}\mathcal{L}^{j}a_{k}$, $\sum_{j,p=1}^{n}\mathcal{L}^{j}\mathcal{L}^{p}\,a_{k}$, $\mathcal{L}^{0}\mathcal{L}^{0}h_{kj}$, $\sum_{j,p=1}^{n}\mathcal{L}^{j}\mathcal{L}^{0}\,h_{kj}$, $\sum_{j,p=1}^{n}\mathcal{L}^{0}\mathcal{L}^{j}\,h_{kj}$, $\sum_{j=1}^{n}\mathcal{L}^{j}\mathcal{L}^{0}a_{k}$,

$$X_{t}^{k}=X_{t_{0}}^{k}+a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{0}+\sum_{j=1}^{n}h_{kj}(t_{0},\mathbb{X}_{t_{0}})I_{j}+\mathcal{L}^{0}a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{00}+\sum_{j=1}^{n}\mathcal{L}^{j}\,a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{j0}$$
$$+\sum_{j=1}^{n}\mathcal{L}^{0}h_{kj}(t_{0},\mathbb{X}_{t_{0}})I_{0j}+\sum_{l,j=1}^{n}\mathcal{L}^{l}\,h_{kj}(t_{0},\mathbb{X}_{t_{0}})I_{lj}$$
$$+\mathcal{L}^{0}\mathcal{L}^{0}a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{000}+\sum_{j=1}^{n}\mathcal{L}^{j}\mathcal{L}^{0}\,a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{j00}$$
$$+\sum_{j=1}^{n}\mathcal{L}^{0}\mathcal{L}^{j}\,a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{0j0}$$
$$+\sum_{j,p=1}^{n}\mathcal{L}^{j}\mathcal{L}^{p}\,a_{k}(t_{0},\mathbb{X}_{t_{0}})I_{jp0}$$

$$+\sum_{j=1}^{n}\mathcal{L}^0\mathcal{L}^0 h_{kj}\left(t_0,\mathbb{X}_{t_0}\right)I_{00j}+\sum_{l,j=1}^{n}\mathcal{L}^l\mathcal{L}^0 h_{kj}\left(t_0,\mathbb{X}_{t_0}\right)I_{l0j}+\sum_{l,j=1}^{n}\mathcal{L}^0\mathcal{L}^l h_{kj}\left(t_0,\mathbb{X}_{t_0}\right)I_{0lj}+\sum_{j,p,l=1}^{n}\mathcal{L}^l\mathcal{L}^p h_{kj}\left(t_0,\mathbb{X}_{t_0}\right)I_{lpj}+R_t,$$

where

$$R_t = I_{0000}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t}+\sum_{j=1}^{n} I_{j000}\,[\mathcal{L}^j\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t}$$
$$+\sum_{j=1}^{n} I_{0j00}\,[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t}+\sum_{j,p=1}^{n} I_{pj00}\,[\mathcal{L}^p\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t}$$
$$+\sum_{j=1}^{n} I_{00j0}\,[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t}+\sum_{j,p=1}^{n} I_{p0j0}\,[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t}$$
$$+\sum_{j,p=1}^{n} I_{0jp0}\,[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t}$$
$$+\sum_{j,p,l=1}^{n} I_{ljp0}\,[\mathcal{L}^l\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t}$$
$$+\sum_{j=1}^{n} I_{000j}\,\left[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 h_{kj}\right]_{t_0,t}+\sum_{j,p=1}^{n} I_{p00j}\,\left[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^0 h_{kj}\right]_{t_0,t}$$
$$+\sum_{j,l=1}^{n} I_{0l0j}\,\left[\mathcal{L}^0\mathcal{L}^l\mathcal{L}^0 h_{kj}\right]_{t_0,t}+\sum_{j,p,l=1}^{n} I_{pl0j}\,\left[\mathcal{L}^p\mathcal{L}^l\mathcal{L}^0 h_{kj}\right]_{t_0,t}$$
$$+\sum_{j=1}^{n} I_{00lj}\,\left[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^l h_{kj}\right]_{t_0,t}$$
$$+\sum_{j,p,l=1}^{n} I_{p0lj}\,\left[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^l h_{kj}\right]_{t_0,t}$$
$$+\sum_{j,p,l=1}^{n} I_{0lpj}\,\left[\mathcal{L}^0\mathcal{L}^l\mathcal{L}^p h_{kj}\right]_{t_0,t}$$
$$+\sum_{j,p,l,r=1}^{n} I_{rlpj}\,\left[\mathcal{L}^r\mathcal{L}^l\mathcal{L}^p h_{kj}\right]_{t_0,t}$$

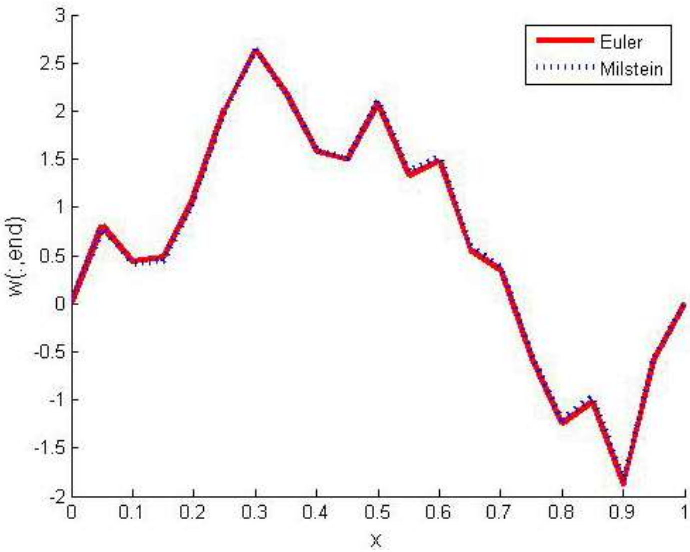
dır.

V. Numerical Application

We consider

$$\begin{cases} du(t,x)=\dfrac{\partial^2}{\partial x^2}u(t,x)dt+dZ_t\,, & t\geq 0,\\ u_t(0)=u_t(1)=0, & t\geq 0,\\ u_0(x)=\sin(\pi x)\,, & 0\leq x\leq 1. \end{cases}$$

We use MATLAB for implementation. In the figure we compare the results.



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