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Lossless Transmission Lines with Time-Varying Specific Parameters

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Abstract — **The present paper is devoted to the investigation of lossless transmission lines with time-varying specific parameters terminated by nonlinear conductive loads with interval of negative conductance. We give a general method for reducing the mixed problem for the arising hyperbolic system to an initial value problem for neutral system on the boundary. Here we overcome difficulties arising from time-varying specific parameters and formulate conditions for the existence of oscillatory solutions.**

Keywords — **transmission lines with time-varying specific parameters, oscillatory solutions, nonlinear boundary value problem, fixed point theorem.**

I. Introduction

The main goal of the present paper is to consider lossless transmission lines with time-varying specific parameters. They are terminated by a nonlinear conductive load with intervals of negative differential conductance (cf. for instance[1]-[3]).

Consider a lossless transmission line, shown on Fig. 1, terminated by a nonlinear load with *V-I* characteristic $i=f(u)=\sum_{n=1}^{m} g_n u^n$, and parallel connected (parasitic) capacitance C_1 , where $E(t)$ is the source function and R_0 − the resistance of the source, $C(x,t)$ – per-unit length capacitance, $L(x,t)$ – per-unit inductance, Λ – the length of the line.

The lossless transmission line is described by the system

$$\frac{\partial i(x,t)}{\partial x}=-\frac{\partial[C(x,t)\,u(x,t)]}{\partial t},\ \frac{\partial u(x,t)}{\partial x}=-\frac{\partial[L(x,t)\,i(x,t)]}{\partial t}. \quad (1)$$

Under the assumptions **(LC):**

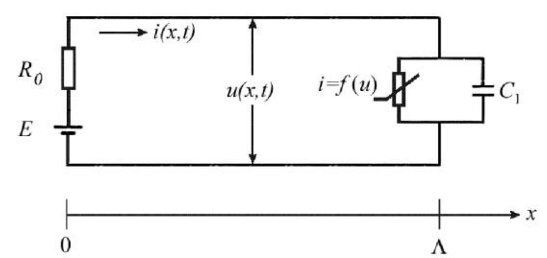

Figure 1. Lossless transmission line with time-varying specific parameters

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$$0<C_0(1-k)\le C(x,t)\le C_0(1+k)\ ;$$

$$0<L_0(1-k)\le L(x,t)\le L_0(1+k)\ ;$$

$$\left|\dot{C}_t(x,t)\right|\le \dot{C}_0;\quad \left|\dot{L}_t(x,t)\right|\le \dot{L}_0$$

where $C_0>0,\ L_0>0,\ 0<k<1,\ \dot{C}_0>0,\ \dot{L}_0>0$ are constants, we formulate a mixed problem for system (1): to find the unknown voltage $u(x,t)$ and current $i(x,t)$ satisfying the system

$$\frac{\partial u(x,t)}{\partial t}+\frac{1}{C(x,t)}\frac{\partial i(x,t)}{\partial x}+\frac{1}{C(x,t)}\frac{\partial C(x,t)}{\partial t}u(x,t)=0$$

$$\frac{\partial i(x,t)}{\partial t}+\frac{1}{L(x,t)}\frac{\partial u(x,t)}{\partial x}+\frac{1}{L(x,t)}\frac{\partial L(x,t)}{\partial t}i(x,t)=0$$

for $(x,t)\in\Pi=\{(x,t)\in R^2:(x,t)\in[0,\Lambda]\times[0,\infty)\}$ with boundary conditions

$$E(t)-u(0,t)=R_0 i(0,t),\quad t\ge 0$$

$$C_1\frac{du(\Lambda,t)}{dt}=i(\Lambda,t)-\sum_{n=1}^{m} g_n u^n(\Lambda,t),\ t\ge 0 \quad (2)$$

and initial conditions

$$u(x,0)=u_0(x), i(x,0)=i_0(x)\quad x\in[0,\Lambda],$$

where $u_0(x),\ i_0(x)$ are prescribed initial functions.

II. Transformation of the Hyperbolic System in Diagonal Form

The system (1) can be rewritten in the matrix form

$$\begin{bmatrix}u_t\\ i_t\end{bmatrix}+\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}\begin{bmatrix}u_x\\ i_x\end{bmatrix}+\begin{bmatrix}C_t/C & 0\\ 0 & L_t/L\end{bmatrix}\begin{bmatrix}u\\ i\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

Via denotations $U=\begin{bmatrix}u\\ i\end{bmatrix},\ A=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix},\ A_1=\begin{bmatrix}C_t/C & 0\\ 0 & L_t/L\end{bmatrix}$ we have

$$U_t+AU_x+A_1U=0\,. \quad (3)$$

To transform $A=\begin{bmatrix}0 & 1/C\\ 1/L & 0\end{bmatrix}$ in diagonal form we solve

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the equation $\begin{vmatrix} -\lambda & 1/C \\ 1/L & -\lambda \end{vmatrix} = 0$ whose roots are

$$\lambda_1(x,t) = 1/\sqrt{L(x,t)C(x,t)}, \ \ \lambda_2(x,t) = -1/\sqrt{L(x,t)C(x,t)} .$$

Eigen-vectors are

$$\left(\xi_1^{(1)}, \xi_2^{(1)}\right) = \left(\sqrt{C}, \sqrt{L}\right), \ \left(\xi_1^{(2)}, \xi_2^{(2)}\right) = \left(-\sqrt{C}, \sqrt{L}\right).$$

Denote by H the matrix formed by eigen-vectors $H(x,t) = \begin{bmatrix} \sqrt{C(x,t)} & \sqrt{L(x,t)} \\ -\sqrt{C(x,t)} & \sqrt{L(x,t)} \end{bmatrix}$. Its inverse one is

$$H^{-1}(x,t) = \begin{bmatrix} 1/\left(2\sqrt{C(x,t)}\right) & -1/\left(2\sqrt{C(x,t)}\right) \\ 1/\left(2\sqrt{L(x,t)}\right) & 1/\left(2\sqrt{L(x,t)}\right) \end{bmatrix}$$ and then

$$HAH^{-1} = \begin{bmatrix} 1/\sqrt{L(x,t)C(x,t)} & 0 \\ 0 & -1/\sqrt{L(x,t)C(x,t)} \end{bmatrix}.$$

Introduce new variables $Z = \begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix}$, where $Z = HU$ and $U = H^{-1}Z$, we have

$$V(x,t) = \sqrt{C(x,t)}\, u(x,t) + \sqrt{L(x,t)}\, i(x,t)$$
$$I(x,t) = -\sqrt{C(x,t)}\, u(x,t) + \sqrt{L(x,t)}\, i(x,t)$$

or

$$u(x,t) = \frac{1}{2\sqrt{C(x,t)}} V(x,t) - \frac{1}{2\sqrt{C(x,t)}} I(x,t)$$
$$i(x,t) = \frac{1}{2\sqrt{L(x,t)}} V(x,t) + \frac{1}{2\sqrt{L(x,t)}} I(x,t).$$

Substituting $U = H^{-1}Z$ in (3) we obtain

$$\frac{\partial\left(H^{-1}Z\right)}{\partial t} + A\frac{\partial\left(H^{-1}Z\right)}{\partial x} + A_1\left(H^{-1}Z\right) = 0$$

and multiplying from the left by H we get

$$\frac{\partial Z}{\partial t} + \left(HAH^{-1}\right)\frac{\partial Z}{\partial x} + \left(H\frac{\partial H^{-1}}{\partial t} + HA\frac{\partial H^{-1}}{\partial x} + HA_1H^{-1}\right)Z = 0 .$$

We reach the system

$$\frac{\partial V(x,t)}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial V(x,t)}{\partial x} + \frac{1}{4}\left(\frac{\partial \ln LC}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial \ln LC}{\partial x}\right)V(x,t) +$$
$$+\frac{1}{4}\left(\frac{\partial \ln(L/C)}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial \ln(L/C)}{\partial x}\right)I(x,t) = 0, \qquad (4)$$
$$\frac{\partial I(x,t)}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial I(x,t)}{\partial x} + \frac{1}{4}\left(\frac{\partial \ln(L/C)}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial \ln(L/C)}{\partial x}\right)V(x,t) +$$
$$+\frac{1}{4}\left(\frac{\partial \ln LC}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial \ln LC}{\partial x}\right)I(x,t) = 0.$$

Recall the denotation $v(x,t) = 1/\sqrt{L(x,t)C(x,t)}$ – the speed of propagation. The system

$$\frac{d\xi}{d\tau} = \frac{1}{\sqrt{L(\xi,\tau)C(\xi,\tau)}}, \ \xi(t) = x$$
$$\frac{d\eta}{d\tau} = -\frac{1}{\sqrt{L(\eta,\tau)C(\eta,\tau)}}, \ \eta(t) = x$$

for each $(x,t) \in \Pi$ has a unique solution. Then

$$T_\xi(t) = \Lambda / v(\xi(t),t) = \Lambda\sqrt{L(\xi(t),t)C(\xi(t),t)} ,$$
$$T_\eta(t) = \Lambda / v(\eta(t),t) = \Lambda\sqrt{L(\eta(t),t)C(\eta(t),t)}$$

with derivatives

$$\dot{T}_\xi(t) = \frac{2\Lambda}{4\sqrt{LC}}\left[\frac{\partial(LC)}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial(LC)}{\partial x}\right],$$
$$\dot{T}_\eta(t) = \frac{2\Lambda}{4\sqrt{LC}}\left[\frac{\partial(LC)}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial(LC)}{\partial x}\right].$$

III. Reducing the Mixed Problem to an Initial Value Problem on the Boundary

For particular case $L = L(t),\ C = C(t)$ we obtain $\frac{\partial \ln LC}{\partial x} = 0$ and $\frac{\partial \ln(L/C)}{\partial x} = 0$. Consequently

$$T_\xi(t) = \Lambda / v(t) = \Lambda\sqrt{L(t)C(t)} = T_\eta(t) \equiv T(t) ,$$
$$\dot{T}(t) = \Lambda\left(\dot{L}(t)C(t) + L(t)\dot{C}(t)\right)/\left(2\sqrt{L(t)C(t)}\right).$$

We make assumptions **(GC):**

$$\frac{d^2 \ln(L(t)C(t))}{dt^2} = 0, \ \ \frac{d \ln(L(t)/C(t))}{dt} = 0 .$$

Denote by $\sigma(t) = \frac{1}{4}\frac{d \ln(L(t)C(t))}{dt}$, $\gamma(t) = \frac{1}{4}\frac{d \ln(L(t)/C(t))}{dt}$.

Therefore $\dot{\sigma}(t) = 0\ (\sigma = \mathrm{const}),\ \gamma(t) = 0 \Rightarrow L(t)/C(t) = \mathrm{const}$ and then (4) can be simplified in the form

$$\frac{\partial V(x,t)}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial V(x,t)}{\partial x} + \sigma V(x,t) = 0$$
$$\frac{\partial I(x,t)}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial I(x,t)}{\partial x} + \sigma I(x,t) = 0. \qquad (5)$$

Let us set $V(x,t) = e^{-\sigma t}W(x,t),\ I(x,t) = e^{-\sigma t}J(x,t)$ and substitute in (5). We get

$$\frac{\partial W(x,t)}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial W(x,t)}{\partial x} = 0, \ \ \frac{\partial J(x,t)}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial J(x,t)}{\partial x} = 0. \quad (6)$$

To obtain new boundary conditions with respect to the new variables we substitute

$$u(x,t)=\frac{e^{-\sigma t}}{2\sqrt{C(t)}}W(x,t)-\frac{e^{-\sigma t}}{2\sqrt{C(t)}}J(x,t),$$

$$i(x,t)=\frac{e^{-\sigma t}}{2\sqrt{L(t)}}W(x,t)+\frac{e^{-\sigma t}}{2\sqrt{L(t)}}J(x,t)$$

into (2) and obtain:

$$E(t)-\frac{1}{2\sqrt{C(t)}}e^{-\sigma t}W(0,t)+\frac{1}{2\sqrt{C(t)}}e^{-\sigma t}J(0,t)=$$

$$=\frac{R_0}{2\sqrt{L(t)}}e^{-\sigma t}W(0,t)+\frac{R_0}{2\sqrt{L(t)}}e^{-\sigma t}J(0,t),\quad t\ge 0,$$

$$C_1\frac{d}{dt}\left(\frac{e^{-\sigma t}W(\Lambda,t)-e^{-\sigma t}J(\Lambda,t)}{2\sqrt{C(t)}}\right)=\frac{e^{-\sigma t}W(\Lambda,t)}{2\sqrt{L(t)}}+$$

$$+\frac{e^{-\sigma t}J(\Lambda,t)}{2\sqrt{L(t)}}-\sum_{n=1}^{m}\frac{g_n}{2^n}\left(\frac{e^{-\sigma t}W(\Lambda,t)-e^{-\sigma t}J(\Lambda,t)}{\sqrt{C(t)}}\right)^n,\ t\ge 0.$$

For the new initial conditions we proceed from

$$W(x,t)=e^{\sigma t}\sqrt{C}\,u(x,t)+e^{\sigma t}\sqrt{L}\,i(x,t),$$

$$J(x,t)=-e^{\sigma t}\sqrt{C}u(x,t)+e^{\sigma t}\sqrt{L}\,i(x,t).$$

Then

$$W(x,0)=\sqrt{C(0)}\ u(x,0)+\sqrt{L(0)}\ i(x,0)=$$

$$=\sqrt{C(0)}\ u_0(x)+\sqrt{L(0)}\ i_0(x)\equiv W_0(x)$$

$$J(x,0)=-\sqrt{C(0)}\ u(x,0)+\sqrt{L(0)}\ i(x,0)=$$

$$=-\sqrt{C(0)}\ u_0(x)+\sqrt{L(0)}\ i_0(x)\equiv J_0(x).$$

So on the base of results from [4] and [5] we formulate the following mixed problem: to solve (6) in $x\in[0,\Lambda];t\in[0,\infty)$ with initial and boundary conditions

$$W(x,0)=W_0(x),\ J(x,0)=J_0(x),\ \ x\in[0,\Lambda].$$

$$2E(t)e^{\sigma t}-\frac{W(0,t)+J(0,t)}{\sqrt{C(t)}}=R_0\frac{W(0,t)+J(0,t)}{\sqrt{L(t)}},\quad t\ge 0$$

$$\frac{d}{dt}\left(\frac{e^{-\sigma t}}{\sqrt{C(t)}}\big(W(\Lambda,t)-J(\Lambda,t)\big)\right)-e^{-\sigma t}\frac{W(\Lambda,t)+J(\Lambda,t)}{C_1\sqrt{L(t)}}=$$

$$=-\sum_{n=1}^{m}\frac{g_n}{2^{n-1}C_1}\left(e^{-\sigma t}\frac{W(\Lambda,t)-J(\Lambda,t)}{\sqrt{C(t)}}\right)^n,\ t\ge 0.$$

After obvious transformations the boundary conditions become

$$W(0,t)=\frac{2e^{\sigma t}E(t)\sqrt{L(t)}}{R_0+Z_0(t)}-J(0,t),\quad t\ge 0,$$

$$\dot J(\Lambda,t)=\dot W(\Lambda,t)-\left(\sigma+\frac{\dot C(t)}{2C(t)}+\frac{\sqrt{C(t)}}{C_1\sqrt{L(t)}}\right)W(\Lambda,t)+$$

$$+\left(\sigma+\frac{\dot C(t)}{2C(t)}-\frac{\sqrt{C(t)}}{C_1\sqrt{L(t)}}\right)J(\Lambda,t)+\frac{1}{C_1}\sum_{n=1}^{m}\frac{g_ne^{-(n-1)\sigma t}\big(W(\Lambda,t)-J(\Lambda,t)\big)^n}{\left(2\sqrt{C(t)}\right)^{n-1}}.$$

Repeating reasoning from [4] and [5] we integrate along the characteristics and obtain

$$W(0,t)=W(\Lambda,t-T(t)),\ J(0,t-T(t))=J(\Lambda,t).$$

Assuming that $W(\Lambda,t)=W(t)$, $J(0,t)=J(t)$ are unknown functions we obtain the system

$$\begin{aligned}&\dot W(t)=\dot J(t-T(t))+\left(\sigma+\frac{\dot C(t)}{2C(t)}+\frac{\sqrt{C(t)}}{C_1\sqrt{L(t)}}\right)W(t)-\\&-\left(\sigma+\frac{\dot C(t)}{2C(t)}-\frac{\sqrt{C(t)}}{C_1\sqrt{L(t)}}\right)J(t-T(t))-\\&-\frac{1}{C_1}\sum_{n=1}^{m}\frac{g_ne^{-(n-1)\sigma t}\big(W(t)-J(t-T(t))\big)^n}{\left(2\sqrt{C(t)}\right)^{n-1}}\equiv F_W\\&J(t)=\frac{2e^{\sigma t}E(t)\sqrt{L(t)}}{R_0+\sqrt{L(t)/C(t)}}-W(t-T(t)),\quad t\ge 0\end{aligned}\tag{7}$$

with delay $T(t)$. In order to formulate a correct problem we should prescribe initial functions on the interval $[-T(0),0]$. This can be made by transition of the first initial functions along characteristics of the hyperbolic system (cf. [5]). The obtained functions we denote by $W_0(t)$ and $J_0(t)$.

IV. Existence-Uniqueness of an Oscillatory Continuous Solution

Now we are able to formulate the main problem: to find a solution of (7) with advanced prescribed zeroes on an interval $[0,\infty)$, where $W_0(t)$ and $J_0(t)$ are prescribed oscillating functions on the interval $[-T(0),0]$.

Let $S_T=\{\tau_l\}_{l=1}^n, n\in N$ be the set of zeroes of the initial function, that is, $W_0(\tau_l)=0,\ J_0(\tau_l)=0$ such that $\tau_1=-T(0),\ \ \tau_n=0$. Besides

$$\max\{\tau_{l+1}-\tau_l: l=0,1,\ldots,n\}\le\sup\{T(t):t\in[0,\infty)\}=T_0<\infty.$$

Let $S=\{t_l\}_{l=0}^{\infty}$ be a strictly increasing sequence of real numbers satisfying the following conditions **(C)**:

(C1) $\lim_{l\to\infty}t_l=\infty$; **(C2)** for every l there is $s<l$ such that $t_l-T(t_l)=t_s$ where $t_s\in S_T\cup S$.

It follows

$$0\le\Delta=\inf\{t_{l+1}-t_l: l=0,1,2,\ldots\}\le\sup\{t_{l+1}-t_l: l=0,1,2,\ldots\}=T_0<\infty$$

Introduce the sets

$$M_W=\left\{W(.)\in C[0,\infty): W(t_l)=0 \text{ and } |W(t)|\le W_0e^{\mu(t-t_l)},\ t\in[t_l,t_{l=1}]\right\}$$

$$M_J=\left\{J(.)\in C[0,\infty): J(t_l)=0 \text{ and } |J(t)|\le J_0e^{\mu(t-t_l)},\ t\in[t_l,t_{l=1}]\right\}$$

$(l=0,1,2,\ldots)$, where W_0,J_0,μ are positive constants, and the following families of pseudo-metrics

$$\rho_\mu^{(l)}(W,\overline W)=\max\left\{e^{-\mu(t-t_l)}\left|W(t)-\overline W(t)\right|: t\in[t_l,t_{l+1}]\right\},$$

$$\rho_\mu^{(l)}(J,\overline J)=\max\left\{e^{-\mu(t-t_l)}\left|J(t)-\overline J(t)\right|: t\in[t_l,t_{l+1}]\right\}.$$

The set $M_W\times M_J$ turns out into a complete uniform space (cf.[5])with respect to the saturated family of pseudo-metrics $\rho_\mu^{(l)}\big((W,J),(\overline W,\overline J)\big)$ $(l=0,1,2,\ldots)$. Using (7) we define the

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operator $B=\left(B_W(W,J),B_W(W,J)\right)$: $M_W\times M_J\to M_W\times M_J$ by the formulas

$$B_W(W,J)(t):=\int_{t_l}^{t}F_W(W,J)(s)ds-\frac{t-t_l}{t_{l+1}-t_l}\int_{t_l}^{t_{l+1}}F_W(W,J)(s)ds,$$

$$B_J(W,J)(t)=\frac{2e^{\sigma t}E(t)\sqrt{L(t)}}{R_0+\sqrt{L(t)/C(t)}}-W(t-T(t)),$$

$t\in[t_l,t_{l+1}]\ (l=0,1,2,\ldots)$.

We call a solution of (6) the solution of the operator equation $(W,J)=\left(B_W(W,J),B_J(W,J)\right)$.

After some preliminary assertions we reach the main result.

Theorem 1. Let the following conditions be fulfilled:

1) The initial functions $W_0(.),J_0(.)\in C^1[-T(0),0]$ satisfy $|W_0(t)|\le W_0e^{\mu(t-\tau_l)},|J_0(t)|\le J_0e^{\mu(t-\tau_l)},t\in[\tau_l,\tau_{l+1}]$;

2) $E(t_k)=0; e^{\sigma t}|E(t)|\le W_0e^{\mu(t-t_l)},\ t\in[t_l,t_{l+1}];\ (W_0=J_0)$;

3) Assumptions **(LC)** and **(GC)** are valid and

$$\left|\dot T(t)\right|\le\frac{\Lambda(1+k)}{2(1-k)}\frac{\dot L_0C_0+L_0\dot C_0}{\sqrt{L_0C_0}}=\dot T_0<1;$$

4) The following inequalities are satisfied

$$\frac{e^{-\mu\Delta}J_0}{1-\dot T_0}+e^{\mu T_0}\frac{W_0+J_0e^{-\mu\Delta}}{\mu}\left(|\sigma|+\frac{\dot C_0}{2C_0(1-\kappa)}+\frac{1}{C_1}\sqrt{\frac{C_0(1+k)}{L_0(1-k)}}+\right.$$
$$\left.+\frac{1}{C_1}\sum_{n=1}^{m}|g_n|\left(\frac{W_0+J_0e^{-\mu\Delta}}{2\sqrt{C_0(1-k)}}\right)^{n-1}\frac{e^{(n-1)\mu T_0}+\ldots+1}{n}\right)\le J_0;$$

$$\frac{2E_0\sqrt{L_0(1+k)}}{R_0+\sqrt{L_0(1-k)/C_0(1+k)}}+W_0e^{-\mu\Delta}\le J_0;$$

$$K_W=\frac{e^{-\mu\Delta}}{1-\dot T_0}+\frac{e^{\mu T_0}\left(e^{-\mu\Delta}+1\right)}{\mu}\left(|\sigma|+\frac{\dot C_0}{2C_0(1-k)}+\frac{1}{C_1}\sqrt{\frac{C_0(1+k)}{L_0(1-k)}}+\right.$$
$$\left.+\sum_{n=1}^{m}\frac{|g_n|\left(e^{(n-1)\mu T_0}+\ldots+1\right)}{C_1}\left(\frac{W_0+J_0e^{-\mu\Delta}}{2\sqrt{C_0(1-k)}}\right)^{n-1}\right)<1;$$

$$K_J=\frac{\left|\sqrt{L_0(1+k)/\left(C_0(1-k)\right)}-R_0\right|}{\sqrt{L_0(1-k)/\left(C_0(1+k)\right)}+R_0}e^{-\mu\Delta}<1.$$

Then there exists a unique oscillatory solution of (6), belonging to $M_W\times M_J$.

V. Numerical Example

Consider a transmission line with specific parameters $L(t)=L_0\left(1+k\cos\theta(t)\right)$, $C(t)=C_0\left(1+k\cos\theta(t)\right)$. Our purpose here is to find the explicit form of $\theta(t)$ in order obtain a "noise-free signal".

We check assumption **(GC).**

Assuming $\dfrac{\ddot L(t)L(t)-\dot L^2(t)}{L^2(t)}=\dfrac{\ddot C(t)C(t)-\dot C^2(t)}{C^2(t)}=0$ we get

$$\frac{d^2\ln\left(L(t)C(t)\right)}{dt^2}=\frac{d}{dt}\left(\frac{\dot L(t)}{L(t)}+\frac{\dot C(t)}{C(t)}\right)=$$
$$=\frac{\ddot L(t)L(t)-\dot L^2(t)}{L^2(t)}+\frac{\ddot C(t)C(t)-\dot C^2(t)}{C^2(t)}=0.$$

For $\theta(t)$ we have

$$\ddot C(t)C(t)-\dot C^2(t)=0\Rightarrow$$
$$C_0\left(1+k\cos\theta(t)\right)C_0\left(-k\sin\theta(t)\ddot\theta(t)-k\cos\theta(t)\dot\theta^2(t)\right)-$$
$$-C_0^2\left(-k\sin\theta(t)\dot\theta(t)\right)^2=0.$$

We put $\dot\theta=p(\theta)\Rightarrow\ \ddot\theta=\dfrac{dp}{d\theta}p$ and $\dfrac{d\theta}{dt}=K_0e^{-\int\frac{k+\cos\theta}{\sin\theta(1+k\cos\theta)}d\theta}$.

As usually we put $\tan\dfrac{\theta}{2}=s\Rightarrow\ \theta=2\arctan s\Rightarrow d\theta=\dfrac{2}{1+s^2}ds$.

Then

$$\frac{d\theta}{dt}=K_0e^{\ln\frac{(1-k)\tan^2(\theta/2)+1+k}{\tan(\theta/2)}}=K_0\frac{(1-k)\tan^2(\theta/2)+1+k}{\tan(\theta/2)}.\quad(8)$$

In view of $\theta_0=\theta(0);\ \dot\theta_0=\dot\theta(0)$ for $t=0$ we have

$$\dot\theta_0=K_0\frac{(1-k)\tan^2(\theta_0/2)+1+k}{\tan(\theta_0/2)}\Rightarrow K_0=\frac{\dot\theta_0\tan(\theta_0/2)}{(1-k)\tan^2(\theta_0/2)+1+k}.$$

It follows

$$\int\frac{\tan(\theta/2)}{(1-k)\tan^2(\theta/2)+1+k}d\theta=K_0t+K_1;$$

$$-\frac{1}{2k}\int\frac{-k\sin\theta}{1+k\cos\theta}d\theta=K_0t+K_1\Rightarrow$$

$$\ln(1+k\cos\theta)=-2kK_0t-2kK_1\Rightarrow K_1=\frac{\ln(1+k\cos\theta_0)}{-2k};$$

$$\cos\theta(t)=\frac{e^{-2k(K_0t+K_1)}-1}{k};\ \theta(t)=\arccos\frac{e^{-2k(K_0t+K_1)}-1}{k}$$

which is valid for $-1\le\dfrac{e^{-2k(K_0t+K_1)}-1}{k}\le1$.

The functions $\left|\dot L(t)\right|\le L_0k\left|\dot\theta(t)\right|\le L_0k\breve{\dot\theta}=\dot L_0$ and $\left|\dot C(t)\right|\le C_0k\left|\dot\theta(t)\right|\le C_0k\breve{\dot\theta}=\dot C_0$ are bounded provided $\dot\theta(t)$ is bounded. In view of the right-hand side of (8) $\dot\theta(t)$ is unbounded for $\theta(t)\to\pi$ and $\theta(t)\to0$ that is, $\left(e^{-2k(K_0t+K_1)}-1\right)/k\to-1$ and $\left(e^{-2k(K_0t+K_1)}-1\right)/k\to1$. Thus we suppose

$-1+\delta\le\cos\theta(t)=\dfrac{e^{-2k(K_0t+K_1)}-1}{k}\le1-\delta$ for sufficiently small $\delta>0$. Since

$$\dot\theta=K_0\frac{(1-k)\tan^2(\theta/2)+1+k}{\tan g(\theta/2)}=$$

$$= K_0\left[(1-k)\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}+(1+k)\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right]$$

we have to substitute in the last right-hand side $\cos\theta(t)$ by $-1+\delta$ and $1-\delta$, and obtain:

$$\dot\theta_1 = K_0\left[(1-k)\sqrt{\frac{2-\delta}{\delta}}+(1+k)\sqrt{\frac{\delta}{2-\delta}}\right];$$

$$\dot\theta_2 = K_0\left[(1-k)\sqrt{\frac{\delta}{2-\delta}}+(1+k)\sqrt{\frac{2-\delta}{\delta}}\right];$$

$$\breve{\dot\theta} = \max\{\dot\theta_1;\dot\theta_2\} = 2|K_0|\frac{1+k(1-\delta)}{\sqrt{2-\delta}\sqrt{\delta}}.$$

In other words if we want to obtain "noise-free" signal we have to choose $\theta(t)=\arccos\left(e^{-2k(K_0t+K_1)}-1\right)/k$ and then

$$\sigma = \frac{-k\dot\theta_0\tan(\dot\theta_0/2)}{(1-k)\tan^2(\dot\theta_0/2)+1+k}.$$

Obviously the sign of σ defines the behavior of the solution amplification or attenuation. The *V-I* characteristic of the nonlinear element

$$f(u)=g_1u+g_1u^2+g_3u^3=0{,}001u-0{,}5u^2+(1/3)u^3$$

has an interval of negative differential conductance.

Let us take $W_0\approx J_0\approx E_0\approx 10^{-11}$; $k=0{,}01$; $\Lambda=1m$; $\mu=5.10^9$; $\delta=0{,}01$; $L_0=0{,}2\mu H/m$; $C_0=5\,pF/m$; $R_0=35\Omega$; $C_1=8.10^{-11}F$.

Then

$$\sqrt{L_0C_0}=10^{-9};\quad \sqrt{L_0/C_0}=\sqrt{0{,}04.10^6}=200\ \Omega;$$

$$(0{,}99)\times10^{-9}\le T(t)=\Lambda\sqrt{L(t)C(t)}\le(1{,}01)\times10^{-9};$$

$$\mu\Delta=4{,}95\Rightarrow e^{\mu\Delta}=e^{4{,}95}=1{,}41\times10^2;$$

$$\mu T_0=5{,}05;\ e^{\mu T_0}=e^{5{,}05}=1{,}56.10^2;$$

$$\dot T_0=\frac{1{,}4.10^{-10}\left|\dot\theta_0\tan(\theta_0/2)\right|}{0{,}99\tan^2(\theta_0/2)+1{,}01}<1 \text{ for sufficiently small } \theta_0.$$

The inequalities from Theorem 1 become

$$\frac{0{,}9.10^{-3}}{35+198}+e^{-4{,}95}\le1;$$

$$7.10^{-3}/\left(1-\frac{1{,}4.10^{-10}\left|\dot\theta_0\tan(\theta_0/2)\right|}{0{,}99\tan^2(\theta_0/2)+1{,}01}\right)+$$

$$+\frac{2{,}6.10^{-9}\left|\dot\theta_0\tan(\theta_0/2)\right|}{0{,}99\tan^2(\theta_0/2)+1{,}01}+0{,}1975+0{,}39+0{,}03443+5{,}265.10^{-5}\le1;$$

$$K_W=\frac{|198-35|}{198+35}e^{-4{,}95}=0{,}00504<1;$$

$$K_J=7.10^{-3}/\left(1-\frac{1{,}4.10^{-10}\left|\dot\theta_0\tan(\theta_0/2)\right|}{0{,}99\tan^2(\theta_0/2)+1{,}01}\right)+$$

$$\frac{2{,}6.10^{-9}\left|\dot\theta_0\tan(\theta_0/2)\right|}{0{,}99\tan^2(\theta_0/2)+1{,}01}+0{,}197+0{,}39+0{,}138+4{,}7.10^{-5}<1.$$

We can choose an initial approximation

$$W^{(0)}(t)=W_0\sin\theta(t),\ J^{(0)}(t)=J_0\cos\theta(t).$$

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