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Geometrical Imperfections Sensitivity of a Four-node Membrane Element Using Perturbation Analysis Method

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***Abstract*— By means of Koiter's perturbation approach, the perturbation equations of a geometrically imperfect four-node membrane element is obtained. Perturbation analysis method, a direct procedure to obtain the non-linear responses of geometrically imperfect structures, is discussed and used to expand responses as a power series in terms of the imperfection amplitude. To obtain the unknown coefficients of expansion, a FEM code specially developed.**

Keywords—imperfection, perturbation analysis method, Koiter's perturbation approach, FEM, membrane element

I. Introduction

Imperfections refers to some differences between the designed and what stands at a certain stage of the life of a structure. Imperfections may affect the intrinsic or geometric parameters. Intrinsic parameters are parameters present in the constitutive relations such as the modulus of elasticity, Poisson's ratio and yield stress. In thin-walled components, the thickness is also included in the constitutive equations. An Important feature of intrinsic parameters is that the equilibrium equations of the system are an explicit function of them [1], [9].

Geometric parameters define the geometrical boundaries of a structure. In many built engineering structures, geometric parameters of the final constructed shape have significant differences with respect to as-designed ones. It is very difficult to build a structure with the exact shape specified in the analysis, and this leads to local distortions in the shape. In final structures, sizes may have defects or the curves that define the boundaries may be constructed with errors. Imperfections in geometric parameters may affect the size, the shape, the position of the middle surface. Because of affecting the definition of the geometry of the structure, all these defects will be named and enclosed in the category of Geometrical Imperfections. If the analysis is carried out by a finite element procedure, these geometrical imperfections affect the strain-displacement matrix, $\tilde{B}$, and the limits of integration of the volume or surface considered. A distinctive feature of these imperfections is that the stiffness matrix is not an explicit function of them [1], [4], [7] and [8].

The most common causes of geometrical imperfections are problems arising during the construction process or the damages that may occur after the completeness of structures.

Attempts to develop a general theory to study the responses of an imperfect structure began with Koiter [3], [5], [6] and [7]. Regardless of the arising way of geometrical imperfections, according to [1], there are three different and general methods to analyze geometrically imperfect structures. Direct analysis, the first method, has the simplest idea for modeling imperfections. By using this method, variations in the geometries of a structure can be taken into account through the local properties of elements. In the equivalent load analysis, the second method, a variation is modeled by means of new loading system acting on the perfect structure. Such technique has the advantage that, the degeneracy is confined to the load vector and does not affect the stiffness matrix. Perturbation analysis, the third method, is based on the perturbation theory. This method has been employed to follow a non-linear path, starting at an equilibrium condition that is known, and using a perturbation parameter to advance along the path.

In this paper we attempt to use the perturbation techniques to model a geometrical imperfect structure. By applying a finite element procedure and perturbation technique, we compute displacements in a geometrically imperfect membrane element. After that, the method is utilized to solve a practical problem and a brief comparison between the perturbation and direct analysis methods is presented.

II. Perturbation Technique

As mentioned before, perturbation technique is based on the perturbation theory. The specific form of solution that is employed in perturbation method is based on the study of the effects that small modifications produce on the response of a system [1], [9]. By using the perturbation parameter and mathematical relations, the response expressions are expanded. Perturbation techniques lead to approximate solutions [4]; however, the solutions take the form of analytical functions. This is why perturbation techniques are analytical approximate solutions, in contrast to numerical methods.

One of the most important advantages of perturbation techniques is that it drives a more explicit solution in terms of effective parameters than numerical solutions. By utilizing this method, a sensitivity expression immediately can be obtained from the solution. But, in comparison with numerical methods, perturbation techniques lead to new matrices. Developing these additional matrices and solving the new equations is a disadvantage of this method.

III. Governing Perturbation Equation

In this section the perturbation equations for an imperfect membrane element is obtained. The main idea of this method is to model geometric imperfections by considering them as initial strains.

Assume a perfect and ideal element. This perfect element has displacements from the reference configuration denoted by $\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}}$. On the other hand, a geometrical imperfection pattern is modeled in terms of the same geometric degrees of freedom employed in $\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}}$, represented by $\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}$. According to [1], the net strains occurring under load can be calculated as those which would be produced by a lateral deflection of $\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}} + \tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}$ minus those which would be produced by a lateral deflection of $\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}}$ alone. As a result

$$\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}} = \tilde{U}^{e} + \tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}} \tag{1}$$

When we consider imperfections, we are dealing with non-linearity. So, we have to consider non-linear relations between strains and displacements. The usual finite element notation in problems with non-linear strain-displacement equations is as equation

$$\tilde{\varepsilon}^{e} = \left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{e}\right)\right)\tilde{U}^{e} \tag{2}$$

Where, $\tilde{B}^{e}_{L}\tilde{U}^{e}$ is the linear part of the strains, and $\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}\right)\tilde{U}^{e}$ is the quadratic part. The matrix $\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}\right)$ is often called large displacements matrix and as the formula shows, it is a function of nodal displacements. All of the displacement vectors, ($\tilde{U}^{e}$, $\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}}$ and $\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}$) induce their strains vector denoted by $\tilde{\varepsilon}^{e}$, $\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Perfect}}$ and $\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Imperfect}}$. By using (2), these strains can be related to their displacements. For vector $\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Perfect}}$, we have

$$\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Perfect}} = \left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}}\right)\right)\tilde{U}^{\mathrm{e}}_{\mathrm{Perfect}} \tag{3}$$

By substituting (1) into (3) we obtain that

$$\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Perfect}} = \left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{e} + \tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}\right)\right)\left(\tilde{U}^{e} + \tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}\right) \tag{4}$$

In the same way we can say that the initial strains due to the imperfections are

$$\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Imperfect}} = \left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}\right)\right)\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}} \tag{5}$$

The net strain due to the applied loads ($\tilde{\varepsilon}^{e}$) can be obtained by omitting the imperfect part ($\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{imperfect}}$) from perfect part ($\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{Perfect}}$) of strains. By assuming that $\tilde{B}^{e}_{NL}\left(\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}\right)\tilde{U}^{e} = \tilde{B}^{e}_{NL}\left(\tilde{U}^{e}\right)\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}$ and just considering the linear terms of displacements $\tilde{U}^{e}$, the term $\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}\right)\tilde{U}^{e}$ would be neglected and (2) changes to

$$\tilde{\varepsilon}^{e} = \left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{\mathrm{e}}_{\mathrm{Imperfect}}\right)\right)\tilde{U}^{e} \tag{6}$$

By using the constitutive matrix, Stresses can be computed as

$$\tilde{\sigma}^{e} = \tilde{D}\tilde{\varepsilon}^{e} = \tilde{D}\left(\tilde{B}^{e}_{L} + \tilde{B}^{e}_{NL}\left(\tilde{U}^{e}_{Imperfect}\right)\right)\tilde{U}^{e} \tag{7}$$

At an element level, by using (6) and (7) the total potential energy ($V^{e}_{0} = \frac{1}{2}\int\left(\tilde{\varepsilon}^{\mathrm{e}}_{\mathrm{p}}\right)^{\mathrm{T}}\tilde{\sigma}^{\mathrm{e}}_{\mathrm{p}}\,dV$) can be computted. Differentiating the potential energy with respect to displacement vector and setting the resultant expression to zero gives the stability path which is called the equilibrium equation.

$$\left(\tilde{K}^{e}_{L} + 2\tilde{K}^{e}_{1} + 4\tilde{K}^{e}_{2}\right)\tilde{U}^{e} - \tilde{P}^{e} = 0 \tag{8}$$

In which

$$\tilde{K}^{e}_{L} = \int \left(\tilde{B}^{e}_{L}\right)^{T}\tilde{D}\tilde{B}^{e}_{L}\,dV \tag{9a}$$

$$\tilde{K}^{e}_{1} = \int \left(\tilde{B}^{e}_{L}\right)^{T}\tilde{D}\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}_{imperfect}\right) + \left(\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}_{imperfect}\right)\right)^{T}\tilde{D}\tilde{B}^{e}_{L}\,dV \tag{9b}$$

$$\tilde{K}^{e}_{2} = \int \left(\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}_{imperfect}\right)\right)^{T}\tilde{D}\tilde{B}^{e}_{NL}\left(\tilde{U}^{e}_{imperfect}\right)\,dV \tag{9c}$$

The first matrix ($\tilde{K}^{e}_{L}$) is the linear stiffness matrix which corresponds to the perfect structure. The next two matrices are linearly dependent on the geometrical imperfection pattern $\tilde{U}^{e}_{imperfect}$. In this paper, it is assumed that the global displacement vector for the complete structure is in the form of

$$\tilde{U}^{st}_{Perfect} = \tilde{U}^{st} + \xi\,\tilde{U}^{st}_{Imperfect} \tag{10}$$

In which the term ξ is the amplitude to scale the imperfection shape pattern ($\tilde{U}^{e}_{Imperfect}$). By assembly of each element stiffness matrix, the stiffness matrix of whole structure can be obtained. Considering the relation (8) and using the stiffness matrix of the whole structure, the equilibrium equation changes to global form as

$$\left(\tilde{K}^{st}_{L} + 2\xi\tilde{K}^{st}_{1} + 4\xi^{2}\tilde{K}^{st}_{2}\right)\tilde{U}^{st} - \tilde{P}^{st} = 0 \tag{11}$$

Assume that the solution of (11) is written as

$$\tilde{U}^{st} = \tilde{U}^{st}_{0} + \xi\,\tilde{U}^{st}_{1} + \xi^{2}\tilde{U}^{st}_{2} + \xi^{3}\tilde{U}^{st}_{3} + \cdots \tag{12}$$

In which $\tilde{U}^{st}_{i}(i = 1,2,3,\ldots)$ can be written as

$$\tilde{U}^{st}_{i} = \frac{1}{i!}\frac{d^{i}\tilde{U}^{st}}{d\xi^{i}} \tag{13}$$

Equation (12) is an expansion of the exact solution of imperfect structure $\tilde{U}^{st}$ around the perfect solution $\tilde{U}^{st}_{0}$. The term ξ is the perturbation parameter and can be an arbitrary

parameter. Because of that, the imperfection amplitude have been chosen and substituted instead of perturbation parameter.

By substituting (12) into (11), the equilibrium equation changes to a polynomial with respect to ξ. By differentiating with respect to the perturbation parameter ξ and setting the equation equal to zero continuesly, a set of linear equations will be obtained which called the perturbation equations. These system of equations is independent of the amplitude parameter ξ. Solving these equations leads to the following relations

Zero order perturbation equation

$$\tilde{U}_0^{st} = \tilde{K}_L^{st^{-1}} \tilde{P} \tag{14}$$

First order perturbation equation

$$\tilde{U}_1^{st} = -\tilde{K}_L^{st^{-1}} (2\tilde{K}_1^{st} \tilde{U}_0^{st}) \tag{15}$$

Second order perturbation equation

$$\tilde{U}_2^{st} = -\tilde{K}_L^{st^{-1}} (2\tilde{K}_1^{st} \tilde{U}_1^{st} + 4\tilde{K}_2^{st} \tilde{U}_0^{st}) \tag{16}$$

And n[th] order perturbation equation

$$\tilde{U}_{\mathrm{n}}^{st} = -\tilde{K}_L^{st^{-1}} (2\tilde{K}_1^{st} \tilde{U}_2^{st} + 4\tilde{K}_2^{st} \tilde{U}_1^{st}) \tag{17}$$

It can be shown that the solution of (11) is a parametric solution and function of ξ.

IV. Evaluation of $\tilde{B}_L^e$ and $\tilde{B}_{NL}^e$

As mentioned above, $\tilde{B}_L^e$ is the same matrix as in linear infinitesimal starin analysis and only $\tilde{B}_{NL}^e(\tilde{U}_{imperfect}^e)$ depends on the displacement. In general, $\tilde{B}_{NL}^e(\tilde{U}_{imperfect}^e)$ is a linear function of displacements [2].

In large-displacement theory of elasticity, quadratic terms of displacements are important. Therefore, the theory of strain will be developed without using the linear approximation of classical elasticity [10]. It can be shown that the relation between the strains of a membrane element and its nodal displacements is

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_x \\ v_y \\ u_y + v_x \end{bmatrix} + \frac{1}{2} \begin{bmatrix} {u_x}^2 + {v_x}^2 \\ {u_y}^2 + {v_y}^2 \\ 2u_x u_y + 2v_x v_y \end{bmatrix} \tag{18}$$

And by using the differentiation operator, it can be written as

$$\varepsilon^e = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \tag{19}$$

$$\frac{1}{2} \begin{bmatrix} u_x & 0 & v_x & 0 \\ 0 & u_y & 0 & v_y \\ u_y & u_x & v_y & v_x \end{bmatrix} \begin{bmatrix} \partial/\partial x & 0 \\ \partial/\partial y & 0 \\ 0 & \partial/\partial x \\ 0 & \partial/\partial y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} =$$

$$\left(\tilde{B}_L^e + \tilde{B}_{NL}^e(\tilde{U}_{\mathrm{Imperfect}}^{\mathrm{e}}) \right) \tilde{U}^e$$

By using (19), the matrices $\tilde{B}_L^e$ and $\tilde{B}_{NL}^e(\tilde{U}_{\mathrm{Imperfect}}^{\mathrm{e}})$ can computed.

V. Verification models

In this part we attempt to do an imperfection sensitivity for a cantilever column as shown in Fig. 1. This structure is modeled with 300 four-node membrane elements with a thicknesses equal to beam width (t=20). An axial load P is applied to one end of the column. As an arbitrary load, P is taken as 10% of the column buckling load.

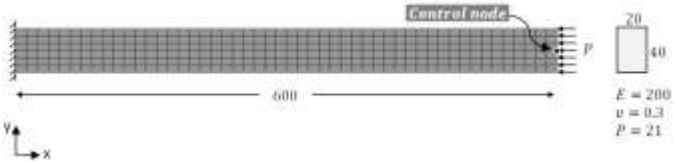

Figure 1. Verification Model Properties.

It is also assumed that the column nodes have an imperfection pattern as shown in Fig. 2. The assumed imperfection pattern for a node with coordinate of $[x_i, y_i]$ has the mathematical relation of $[0, \xi \left(1 - cos\left(\frac{\pi x_i}{2L}\right)\right)]$ in which L is equal to column length ($L = 600$), and the imperfection is a linear amplitude of ξ. In the other hands, the matrix of imperfection pattern would be

$$\tilde{U}_{Imperfect}^e = \begin{bmatrix} 0 \\ \left(1 - cos\left(\frac{\pi x_1}{2 \times 600}\right)\right) \\ \vdots \\ 0 \\ \left(1 - cos\left(\frac{\pi x_{300}}{2 \times 600}\right)\right) \end{bmatrix}_{2NNODE \times 1} \tag{20}$$

In which NNODE is the number of nodes. It is well known that even small to moderate levels of imperfections can significantly affect the behavior of slender structures. Small imperfections cannot change the mechanical characteristics of the structure, but it affects the responses. In this example, the aim is to find a relation between horizontal and vertical displacement of control node, u_{CN} and v_{CN} respectively, with the amplitude of imperfection, ξ, while the amplitude varies from -1 to 1.

Figure 2. Imperfection pattern.

The above example is solved by both direct and perturbation analysis method. In the perturbation analysis method, we only need to solve one model. Then we can plot all mentioned sensitivity graphs using the achieved solution. In direct analysis method, for each amount of Imperfection

amplitude, we have to consider new coordinates for each node and as a result, we have to model and solve several different models. By using a FORTRAN program specially developed in this study, matrices $\widetilde{K}_L^e, \widetilde{K}_1^e$ and $\widetilde{K}_2^e$ of each four-node element have been created and then the global matrices $\widetilde{K}_L^{st}, \widetilde{K}_1^{st}$ and $\widetilde{K}_2^{st}$ have been assembled. The solving procedure of this program is that, by using the matrix $\widetilde{K}_L^{st}$ and (14), matrix $\widetilde{U}_0^{st}$, the perfect system response, should be calculated and after that, by using (15), (16) and (17), matrices $\widetilde{U}_1^{st}, \widetilde{U}_2^{st}$ and $\widetilde{U}_3^{st}$ can be developed. In mentioned program, step value for imperfection amplitude, ξ, can be chosen arbitrarily. The direct analysis responses have been obtained using the ABAQUS/Standard-Explicit 6.10 software and a M3D4, a four-node quadrilateral membrane, element. Fig. 3 and Fig. 4 illustrate the solution of this problem.

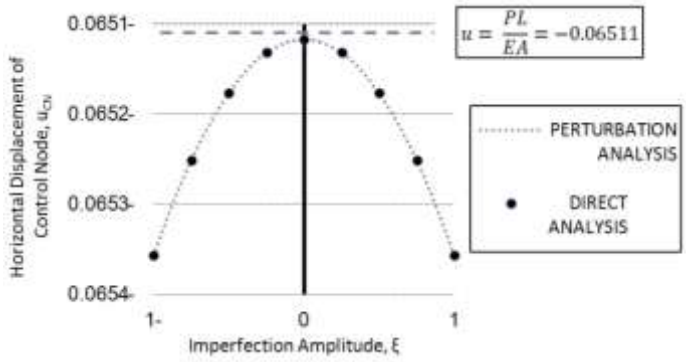

Figure 3. Analysis results - horizontal displacement

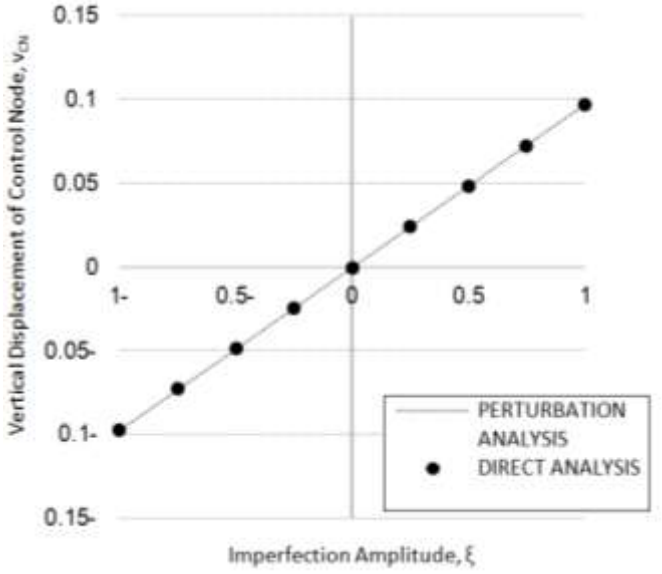

Figure 4. Analysis results - vertical displacement

The results indicate that the horizontal displacement of the control node, u_{CN}, is not affected by the sign of the initial nodal imperfection. But vertical displacement of control node, v_{CN}, represents a growth of the initial nodal imperfection.

As illustrated in Fig. 3, when the imperfection amplitude is equal to zero, both methods show the exact theoretical solution ($\boldsymbol{u_{CN}} = \frac{\boldsymbol{PL}}{\boldsymbol{EA}}$ and $\boldsymbol{v_{CN}} = \boldsymbol{0}$).

VI. Conclusion

Direct analysis is extremely expensive for both hardware needs and computational times. Perturbation technique is much faster than direct analysis. This is a consequence of the fact that by changing the imperfection amplitude, direct analysis needs re-analysis; whereas perturbation technique only requires little additional computational cost.

Perturbation technique and its global approach allows the imperfection sensitivity analysis be performed efficiently.

There are two advantages of the presented procedure. First, in comparison to other numerical solutions, perturbation technique obtains a more explicit solution in terms of effective parameters. Second, the technique can be employed for any order of perturbation desired.

References

[1] L. A. Godoy, Structures with structural imperfections-analysis and behaviour, 1st ed., Great Britain: Pergamon, 1996.

[2] O. C. Zienkiewicz, The finite element method in engineering science, 2nd ed., Great Britain: MacGraw-Hill, 1971.

[3] R. Casciaro, G. Garcea, G. Attanasio, and F. Giordano, "Perturbation approach to elastic post-buckling analysis," Computers & Structures J., vol. 66, No. 5, pp. 585-595, 1997.

[4] L. A. Godoy, "Stresses and pressures in thin-walled structures with damage and imperfections," Thin-walled Structures J., vol. 32, pp. 181-206, 1998.

[5] A. D. Lanzo, "A Koiter's perturbation strategy for the imperfection sensitivity analysis of thin-walled structures with residual stresses," Thin-walled Structures J., vol. 37, pp. 77-95, 2000.

[6] C. A. Featherston, "Imperfection sensitivity of flat plates under combined compression and shear," Non-Linear Mechanics J., vol. 36, pp. 249-259, 2001.

[7] C. A. Schenk, G. I. Schueller, "Buckling analysis of cylindrical shells with random geometric imperfections," Numerical Methods in Engineering J., vol. 52, pp. 1465-1485, 2001.

[8] L. A. Godoy, and E. G. Banchio, "Singular perturbations for sensitivity analysis in symmetric bifurcation buckling," Non-Linear Mechanics J., vol. 38, pp. 1119-1132, 2003.

[9] Z. R. Tahir, P. Mandal, "A new perturbation technique in numerical study on buckling of composite shells under axial compression," World Academy of Science, Engineering and Technology J., vol. 70, pp. 77-86, 2012.

[10] A. P. Boresi, K. P. Chong, and J. D. Lee, Elasticity in engineering mechanics, 3rd ed., Hoboken, New Jersey: John Wiley & Sons, 2011.