The multi pump absorption model for the containment of toxic gases in a tunnel.

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Abstract— In this work we study the diffusion of a toxic gas in a tunnel due to an explosion. The aim is to to realize a control system for identification and immediate containment procedures. To this end, many aspiration pumps are turned on to mitigate the effects. The model utilizes the diffusion partial differential equation with non-homogeneous terms for the aspiration pumps.

Keywords: Analytical solution · Circular tunnel · Explosion · Toxic gas · Dispersion

I. Introduction

In a previous paper [1] the diffusions of toxis gases in a tunnel was studied to better design a mechanism of containment. Our study analyzed the diffusion mechanism in detail. In this contribution, this result is generalized by using a Laplace transform mechanism.

The project is interesting for security reasons and for control of terrorist events. This study is intended as a mechanism for decision support and consider. In a first instance, some aspects of the events occurred in the terrorist attack on the Tokyo subway in 1995. We wish to to try to identify some useful elements for prevention, containment and development of specific intervention procedures. The purpose of the research is to study the potential behaviors of a possible neurotoxin chemical agent diffusion (such as a nerve gas) in a public place (a subway), to monitor and define the propagation models. We wish to identify certain constants and parameters that inserted in complex models may allow precise monitoring of diffusion processes for toxic gases propagation in a tunnel.

Usually the detection of contamination is carried out in large part to events always present. Active network, which continuously detects air quality, are rare and expensive. The results of the study may be useful to design systems with fixed and mobile sensors networks, to monitor and allow rapid detection, risk assessment, and immediate detection of procedures for containment and remediation.

The paper examines so the spread of a toxic agent in order to realize a control system— via a wireless network of active sensors—for identification and immediate containment procedures.

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Because of the explosion in the tunnel, we suppose that one or more aspiration pumps is turned on to mitigate the effects of the terroristic act. Unlike the case of systems described by ordinary differential equations, whose control theory has come to an advanced level of understanding, the nature of the infinite-dimensional systems described by PDE makes difficult to solve control problems and estimates identification for this class of systems.

II. The Single Pump Model and the Analytical Solution

We consider a tunnel with a z-coordinate chosen along its axis. Cylindrical coordinate are chosen and r, θ span the transversal cylindrical region. We suppose that, at $t = 0$ in the origin, an explosion of toxic gas occurs. At b meters on the right of the explosion point, we suppose that a aspiration pump is present and an electronic device immediately turns it on. We assume that the dependence on θ is not relevant, and so, we consider an axisymmetric problem; so doing we denote with $C(r, z, t)$ the concentration of the toxic gas generated by the explosion. This working hypothesis does not take into account of some effects: e.g. real geometry of the tunnel, weight, humidity. We assume that the concentration will obey the following partial differential equation [2,3,4]:

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} - \frac{1}{D}\frac{\partial C}{\partial t} = pC(r,b,t)\delta(r)\frac{\delta(z-b)}{r}$$

where D is the diffusion coefficient. We assume also that the radial coordinate r verifies $0 \leq r \leq R$, where R is the tunnel radius and $-\infty < z < \infty$ [1-3].

The right term of this equation models an aspiration pump placed in z = b, r = 0. A coefficient p is introduced to consider the real aspiration power of the pump while generalized δ Dirac functions in z = b, r = 0 are used to model the idea that, in this point, there is a negative source term whose intensity effect is proportional to the local concentration already there present.

Initial and boundary conditions have to be set in order to solve this partial differential equation. To this end, we assume the following "initial condition"

$$\lim_{t\to 0+} C(r,z,t) = \frac{Q\delta(z)\delta(r)}{r}$$

to model the initial explosion in z = 0, r = 0.

Concerning boundary conditions, we assume, null flux across the tunnel boundary in r = R:

$$\frac{\partial C(R,z,t)}{\partial r} = 0$$

In this boundary region, our problem is of the Neumann kind. Concerning asymptotic requirement per "large" z, we assume

$$\lim_{z\to\pm\infty} C(r,z,t) = 0$$

A discussion on the radial dependence in now required. Since Neumann boundary condition holds for r = R, it is obvious that the radial dependence is now assumed. However, one can ask if the simplifying hypothesis C(r, z, t) = C(z, t) is possible. To this end, we study firstly the well know problem when no absorption pumps are present. In this case, we quickly consider equation

$$\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial z^2} - \frac{1}{D}\frac{\partial C}{\partial t} = 0$$

with the same initial conditions. We firstly use a cosine Fourier transform

$$C(r,z,t) = \int_{-\infty}^{+\infty} \cos 2\pi zq) A(q,r,t) dq$$

and then a Dini Bessel expansion [5,6]. In this way, we are able to get the final solution satisfying the correct boundary and initial data. It is:

$$C(r,z,t) = \frac{C^{(0)}(z,t)}{\pi R^2}\left(1 + \sum_{k=1}^{\infty} \frac{J_0\left(\frac{z_k^1 r}{R}\right)}{J_0^2(z_k^1)} e^{-\left(\frac{z_k^1}{R}\right)^2 Dt}\right)$$

Where the "free" solution is

$$C^{(0)}(z,t) = \frac{Q}{2\sqrt{\pi D t}} e^{-\frac{z^2}{4Dt}}$$

In this expression, we have denoted with $J_s(x)$ the s-th Bessel function of the first kind and with $z_k^{(s)}$ the –zeros of this functions. In this expression s=0.

By observing this equation, one recognizes that the effect of taking into account the radial dependence is simply the introduction of an additive "modulation factor" expansion

$$\sum_{k=1}^{\infty} \frac{J_0\left(\frac{z_k^1 r}{R}\right)}{J_0^2(z_k^1)} e^{-\left(\frac{z_k^1}{R}\right)^2 Dt}$$

Each term of this expansion has null integral on the section 0 ≤ r ≤ R. Even if the radial dependence is not negligible, we wish to study the medium quantity

$$C^{(0)}(z,t) = 2\pi \int_{-\infty}^{+\infty} r\, C(r,z,t)\, dr$$

and concentrate our attention on the role of the aspiration pumps. For this motivation, we assume a model where the concentration depends only on z and t and so we ignore the radial dependence.

In a radial framework, we consider the following partial differential equation:

$$\frac{\partial^2 C(z,t)}{\partial z^2} - \frac{1}{D}\frac{\partial C(z,t)}{\partial t} = pC(b,t)\delta(z-b)$$

With the boundary condition

$$\lim_{z\to\pm\infty} C(z,t) = 0$$

and initial data

$$\lim_{t\to 0+} C(z,t) = Q\delta(z)$$

Again, we use Fourier transform technique:

$$C(z,t) = \int_{-\infty}^{+\infty} e^{2\pi i z q}\, a(q,t) dq$$

and we get the following differential equation for a(q,t) :

$$\frac{\partial a(q,t)}{\partial t} + 4\pi^2 q^2 D\, a(q,t) = -pD\, C(b,t) e^{2\pi i b q}$$

This equation is linear and can be solved in a standard way. By substituting the solution in the Fourier transform, one get, after some manipulation:

$$C(z,t) = C^{(0)}(z,t) - \frac{p\sqrt{D}}{2\sqrt{\pi}} \int_0^t e^{\frac{(b-z)^2}{4D(s-t)}} \frac{C(b,s)}{\sqrt{t-s}} ds$$

This equation is the generalized Abel equation of the second kind. Its solution is present in the literature [7]. Its solution, that now we denote with $C^{(1)}(z,t)$ - since, at this stage, there is only one aspiration pump - is:

$$C^{(1)}(z,t) = C^{(0)}(z,t) - Q\, F(|z-b|, p)$$

where we have set:

$$F(z,p) = \frac{p}{4}\ erfc\left(\frac{|z| + b + pDt}{2\sqrt{Dt}}\right) e^{\frac{p}{4}(\,pDt + 2\,|z| + 2b\,)}$$

The term F(z,p) is the correction term due to the aspiration pump, with respect to the free solution. In this expression we have denote with erfc(x) the complementary error function [5].

III. The Multi Pumps Model

In this section we suppose that many aspiration pumps, of equal power p, are used to mitigate the diffusion of the toxic gas.
We consider therefore the following partial differential equation:

$$\frac{\partial^2 C(z,t)}{\partial z^2} - \frac{1}{D}\frac{\partial C(z,t)}{\partial t} = \sum_{k=1}^{\infty} pC(b_k,t)\delta(z-b_k)$$

Concerning boundary and initial conditions, we assume the same ones of the previous section.
If we repeat the same technique of the previous section, we arrive to the following integral equation

$$C(z,t) = C^{(0)}(z,t) - \frac{p\sqrt{D}}{2\sqrt{\pi}}\sum_{k=1}^{\infty}\int_0^t e^{\frac{(b_k-z)^2}{4D(s-t)}}\frac{C(b_k,s)}{\sqrt{t-s}}ds$$

This equation is a generalization of the Abel equation discussed in the previous section. Unfortunately, no known solution of this equation is present in the literature.
For this motivation, in this contribution, we start studying the case of only two aspiration pumps placed in z=b and z=u, respectively by adopting a Laplace transform technique. We set

$$T(z,s) = \int_0^{\infty} C(z,t)\, e^{-st}\, dt$$

and, so doing, we get:

$$T(z,s) = -\frac{p\sqrt{D}}{2\sqrt{s}}\left(T(b,s)e^{-\frac{\sqrt{s}|z-b|}{\sqrt{D}}} + T(u,s)e^{-\frac{\sqrt{s}|z-u|}{\sqrt{D}}}\right) + \frac{Qe^{-\frac{\sqrt{s}|z|}{\sqrt{D}}}}{\sqrt{Ds}}$$

In this equation, the control terms in z=b and z=u are explicitly present. Now by substituting z=b and z=u, we get a system of equation in which the "control close loop" is exactly open. This system reads:

$$T(b,s) = -\frac{p\,D\,\,T(u,s)}{pD+2\sqrt{Ds}}\,e^{\frac{\sqrt{s}\,(b-u)}{\sqrt{D}}} + \frac{Qe^{-\frac{\sqrt{s}b}{\sqrt{D}}}}{pD+2\sqrt{Ds}}$$

$$T(u,s) = -\frac{p\,D\,\,T(b,s)}{pD+2\sqrt{Ds}}\,e^{\frac{\sqrt{s}\,(b-u)}{\sqrt{D}}} + \frac{Qe^{-\frac{\sqrt{s}u}{\sqrt{D}}}}{pD+2\sqrt{Ds}}$$

Now we solve and get

$$T(b,s) = \frac{Q\left(pD\,e^{\frac{\sqrt{s}\,(b-2u)}{\sqrt{D}}} - e^{-\frac{\sqrt{s}b}{\sqrt{D}}}\,(pD+2\sqrt{Ds})\right)}{D\left(Dp^2e^{\frac{\sqrt{s}\,(b-u)}{\sqrt{D}}} - Dp^2 - 4s - 4p\sqrt{Ds}\right)}$$

$$T(u,s) = \frac{2Qe^{-\frac{\sqrt{s}u}{\sqrt{D}}}\sqrt{s}}{D\left(Dp^2e^{\frac{2\sqrt{s}\,(b-u)}{\sqrt{D}}} - Dp^2 - 4s - 4p\sqrt{Ds}\right)}$$

These equations furnish the Laplace transform of the required concentration in the correspondence with the aspiration pumps.

Now the inverse Laplace transform can be calculated. We perform this calculation in a numerical way to obtain the possibility to present a plot. We compare the concentration in the point z=u, in two different models: in the first case only a single aspiration pump in z=b is present. In the second case two aspiration pumps are present, in z=b and z=u respectively (recall that we assumed u>b). One can obtain an estimate of the absorbion process and a reduction, in the plot, of about 40% on the presence of toxic gas.
Some plots and numerical evidence suggest that this behavior is present also for a greater numbers of aspiration pumps.
This research will be further developed to better design a control mechanism for the mitigation process. We think that an automated sensor mechanism can be useful to protect sensible are from terroristic acts. This includes intelligent agents based on IA-CGF (Intelligent Agent Computer Generated Forces) [8,9].

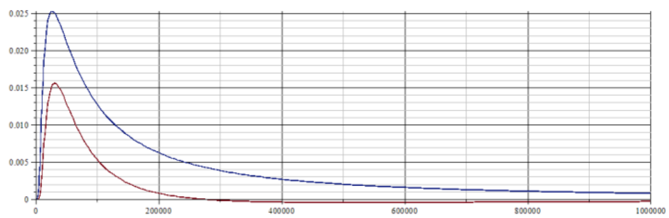

Figure 1. The diffusion of a toxic gas in the point z=u, with and a without a second aspiration ump in z=u.

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