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A New Note on Summability of Factored Infinite Series

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***Abstract*—In this paper, two main theorem dealing with infinite and factored Fourier series, which generalizes some known results, has been generalized to $|A, p_n;\delta|_k$ summability method. This new theorem also includes several known and new results.**

Keywords— ***Summability factors, absolute matrix summability, Fourier series, infinite series, Hölder inequality, Minkowski inequality.***

I. Introduction

Definition 1. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to\infty,\ n\to\infty,\ (P_{-i}=p_{-i}=0,\ i\geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$\sigma_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

Definition 2. The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k\geq 1$ if (see [3])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\Delta\sigma_{n-1}\right|^k<\infty. \tag{3}$$

In the special case when $p_n=1$ for all values of n (resp. $k=1$), $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C,1|_k$ (resp. $\left|\bar{N}, p_n\right|$) summability (see [10]).

Definition 3. The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n;\delta\right|_k$, $k\geq 1$ and $\delta\geq 0$ if (see [7])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\Delta\sigma_{n-1}\right|^k<\infty. \tag{4}$$

where $\Delta\sigma_{n-1}=-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v,\ n\geq 1.$

In the special case, when $p_n=1$ for all values of n (resp. $\delta=0$), $\left|\bar{N}, p_n;\delta\right|_k$ summability is the same as $|C,1;\delta|_k$ (resp. $\left|\bar{N}, p_n\right|_k$) summability.

II. Known Results

Many works dealing with some absolute summability methods of infinite and Fourier series have been done (see [1-2], [4-8], [12-15],[21]). Among them, in [16] Özarslan has proved the following theorem.

Theorem 1. Let $k\geq 1$. If the sequence (s_n) is bounded and the sequences (λ_n) and (p_n) satisfy the following conditions

$$\sum_{n=1}^{m} p_n|\lambda_n|=O(1)\quad as\quad n\to\infty, \tag{5}$$

$$\sum_{n=1}^{m} p_n|\Delta\lambda_n|=O(1)\quad as\quad n\to\infty, \tag{6}$$

$$p_{n+1}=O(p_n), \tag{7}$$

then the series $\sum a_n\lambda_n P_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k\geq 1$.

An Application of Absolute Matrix Summability to Fourier Series

Let $f(x)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The Fourier series of $f(x)$ is

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$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} C_n(x) \tag{8}$$

where (a_n) and (b_n) denote the Fourier coefficients.

The convergence of Fourier series can be ensured by local hypothesis, that is to say, the behavior of the convergence of Fourier series for a particular value of x depends on the behavior of the function in the immediate neighbourhood of this point only (see [20]).

Theorem 2. ([16]) Let $k \geq 1$.The summability $\left|\bar{N}, p_n\right|_k$ of the series $\sum C_n(x)\lambda_n P_n$ at a point is a local property of a generating function if the conditions (5) and (6) are satisfied.

Definition 4. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0,1,... \tag{9}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$if (see [19])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \tag{10}$$

and it is said to be summable $\left|A, p_n\right|_k$, $k \geq 1$, if (see [18])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty \tag{11}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s). \tag{12}$$

If we take $p_n = 1$ for all values of n, $\left|A, p_n\right|_k$ summability is the same as $|A|_k$ summability. Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then $\left|A, p_n\right|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.

and also it is said to be summable $\left|A, p_n; \delta\right|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [17])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left|\bar{\Delta} A_n(s)\right|^k < \infty. \tag{13}$$

III. Main Results

The aim of this paper is to prove a more general theorem which includes some of the above-mentioned result as a special cases.

Theorem 3. Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (s_n) be a bounded sequence and suppose that $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{n0} = 1, \quad n = 0,1,..., \tag{14}$$

$$a_{n-1,v} \geq a_{nv} \quad \text{for} \quad 1 \leq v \leq n-1, \tag{15}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{16}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k - 1}\right\} \quad as \quad m \to \infty, \tag{17}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \quad m \to \infty. \tag{18}$$

If a sequence (λ_n) and (p_n) holds the following conditions,

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} p_n |\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{19}$$

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k} p_n |\Delta\lambda_n| = O(1) \; as \quad m \to \infty, \tag{20}$$

$$p_{n+1} = O(p_n), \tag{21}$$

then the series $\sum a_n \lambda_n P_n$ is summable $|A, p_n; \delta|_k$.

Theorem 4. Let $k \geq 1$ and $0 \leq \delta < 1/k$. The summability $|A, p_n; \delta|_k$ of the series $\sum C_n(x)\lambda_n P_n$ at a point is a local property of a generating function if all the conditions of Theorem 3 are satisfied.

We need the following lemma for the proof of our theorem.

Lemma 5 (see [16]) If the sequences (λ_n) and (p_n) satisfy the conditions (5) and (6) of Theorem 1, then $P_m |\lambda_m| = O(1)$ as $m \to \infty$.

Proof of Theorem 3

(I_n) denotes the A-transform of the series $\sum a_n \lambda_n P_n$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v P_v.$$

Applying Abel's transformation to this sum, we get that

$$\bar{\Delta} I_n(x) = \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv} \lambda_v) P_v \sum_{r=1}^{v} a_r + \hat{a}_{nn} \lambda_n P_n \sum_{v=1}^{n} a_v$$

$$= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv} \lambda_v P_v) + a_{nn} \lambda_n P_n s_n$$

$$= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv}) \lambda_v P_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \lambda_v P_v s_v$$

$$- \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} p_{v+1} s_v + a_{nn} \lambda_n P_n s_n$$

$$= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.$$

To complete the proof of Theorem 3, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n / p_n)^{\delta k + k - 1} |I_{n,r}(x)|^k < \infty, \text{ for } r = 1,2,3,4.$$

Proof of Theorem 4

The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 4 is a consequence of Theorem 3 and Lemma 5 (see [9]).

IV. Conclusions

Corollary 1. If we take $\delta = 0$ in Theorem 3, then we obtain Theorem 1 dealing with $|A, p_n|_k$ summability.

Corollary 2. If we take $a_{nv} = \dfrac{p_v}{P_n}$ in Theorem 3, then we obtain a new theorem concerning with $|\bar{N}, p_n; \delta|_k$ summability factors of Fourier series.

Corollary 3. If we take $a_{nv} = \dfrac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3, then we get a result concerning $|C,1;\delta|_k$ summability factors of Fourier series (see [10]).

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