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Dynamics on a two-prey and one-predator system with hybrid functional responses

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Abstract—— **In this paper, to depict the ecological world realistically, we adopt a two preys and one predator system with hybrid types of functional response, Holling type and Beddington-DeAngelis type. In fact, we show that this system is dissipative and find a necessary condition for the existence of the predator species. In addition, conditions for the persistence of the system are found. And we give a numerical example to attest to our theoretical results. Furthermore, the existence of chaotic phenomena is illustrated by bifurcation diagrams of the system..**

Keywords— predator-prey system, Holling type, Beddington-DeAngelis type, dissipative system, persistence, bifurcation diagram.

I. Introduction

The classical ecological models of interacting populations typically have focused on two-species continuous time systems with one predator and one prey. Many researchers have studied about classical two-species continuous time systems for several functional responses such as Holling-Tanner type ([3,5,16]), Beddington-DeAngelis type ([6,12]), ratio-dependent type([1,7]) and so on. However, it has been recognized that such classical models with two-species can describe only a small number of the phenomena that are commonly observed in nature. For the reason, in recent years, many authors ([9,10,13,14,17,18]) have changed their concerns to the ecological models with three and more species. There are many ways to depict mathematically the changes in population of multi-species. One of basic idea of describing such phenomena is to consider another prey(x_2) different from the prey x_1. Moreover, because the prey x_2 is a different species from the prey x_1, a different functional response is needed to describe the relationship between the prey x_2 and the predator y([2,15]). In fact, if one takes into consideration the handling time of the predator to capture the prey, one figures out that the predator has a Holling type-II functional response([11]) and if one considers the competitions of predators each other to catch the prey, Beddington-DeAngelis type functional response should be adopted([6,12]). Thus, in this paper, we consider the following a hybrid system with two-preys and one-predator.

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\Big(a_1 - b_1x_1(t) - \frac{y(t)}{c+x_1(t)}\Big), \\ \frac{dx_2(t)}{dt} = x_2(t)\Big(a_2 - b_2x_2(t) - \frac{y(t)}{\alpha+x_2(t)+\beta y(t)}\Big), \\ \frac{dy(t)}{dt} = y(t)\Big(-d + \frac{e_1x_1(t)}{c+x_1(t)} + \frac{e_2x_2(t)}{\alpha+x_2(t)+\beta y(t)}\Big). \end{cases} \quad (1)$$

Many researchers([9,10,14,15,17]) have concerned about ecological systems with three and more species like system (1) to understand complex dynamical behaviors of ecological systems in the real world. Especially, the authors in [15] dealt with a prey-predator model with two-type functional response and impulsive biological control and investigated the existence of an asymptotically stable prey-free periodic solution under some conditions and established the permanence conditions for the impulsive system. However, they did not study dynamical properties of the system without impulsive control strategy

II. Main Results

Obviously, the functions in the right hand sides of system (1) are continuous and have continuous partial derivatives on the state space $R^3_+ = \{(x,y,z)^T \mid x \geq 0,\ y \geq 0,\ z \geq 0\}$. Indeed, straightforward computation yields that they are Lipschizian on R^3_+. Thus the solution of system (1) with non-negative initial condition exists and is unique, as the solution of system (1) initiating in the non-negative octant is bounded. Moreover, it is easy to see that if $x_1(0) > 0$, then $x_1(t) > 0$ for all $t > 0$. Same is true for x_2 and y components. Therefore, we conclude clearly that the first octant R^3_+ is an invariant domain of system (1).

A system is said to be *dissipative* if all population initiating in R^3_+ are uniformly limited by their environment [7]. In fact, we have shown that system (1) is dissipative in the following theorem.

Theorem 2.1 *System (1) is dissipative.*

Proof. Since $dx_i(t)/dt \leq x_i(t)(a_i - b_ix_i(t))$, i = 1,2, we have $\limsup_{t\to\infty} x_i(t) \leq a_i/b$, i=1,2. Define $V(t) = e_1 x_1(t) + e_2 x_2(t) + y(t)$. Then $dV(t)/dt \leq a_1(a_1+1)e_1/b_1 + a_2(a_2+1)e_2/b_2 - m V(t)$, where m = min {1,d}. So, by comparison theorem, we obtain that $V(t) \leq M/m - (Me^{-mt})/m$ for $t \geq 0$, where $M = a_1(a_1+1)e_1/b_1 + a_2(a_2+1)e_2/b_2$. Thus $e_1x_1(t) + e_2 x_2(t) + y(t) \leq M/m$ for sufficiently large t, which means that all species are uniformly bounded for any initial value in R^3_+. Therefore, system (1) is dissipative. ■

Theorem 2.2 *A necessary condition for the predator species y to survive is*

$$d < \frac{a_1e_1}{b_1c} + \frac{a_2e_2}{\alpha b_2}.$$

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(2)

Proof. From, the third equation of system (1), we get

$$\begin{aligned}\frac{dy(t)}{dt} &= y(t)\Big(-d+\frac{e_1x_1(t)}{c+x_1(t)}+\frac{e_2x_2(t)}{\alpha+x_2(t)+\beta y(t)}\Big)\\ &\le y(t)(-d+\frac{e_1}{c}x_1(t)+\frac{e_2}{\alpha}x_2(t))\\ &\le y(t)(-d+\frac{a_1e_1}{b_1c}+\frac{a_2e_2}{b_2\alpha})\ \text{(By using Theorem 2.1).}\end{aligned} \qquad (3)$$

Then we have $y(t) \le y_0 e^{At}$, where $A = -d + a_1e_1/(b_1c)+ a_2e_2/(b_2\alpha)$. Thus for $A < 0$, $\lim_{t\to\infty} y(t)= 0$. Hence, the condition (2) is a necessary condition for the survival of the predator y. ■

The Kolmogorov theorem assumes that the existence of either a stable equilibrium point or stable limit cycle behavior in the positive quadrant of phase space of a two-dimensional (2D) dynamical system, provided certain conditions are satisfied ([8]).

$$\begin{cases}\dfrac{dx_1(t)}{dt}=x_1(t)(a_1-b_1x_1(t))-\dfrac{x_1(t)y(t)}{c+x_1(t)},\\ \dfrac{dy(t)}{dt}=y(t)\Big(-d+\dfrac{e_1x_1(t)}{c+x_1(t)}\Big).\end{cases} \qquad (4)$$

The subsystem (4) is a Kolmogorov system under the condition

$$0<\frac{cd}{e_1-d}<\frac{a_1}{b_1}. \qquad (5)$$

Throughout this paper, we assume that the subsystem (4) satisfies the condition (5). By applying the local stability analysis ([4]) to a Kolmogorov system (4) we have the following results;

(1) The equilibrium point $E_{10}=(0,0)$ always exists and is a saddle point.

(2) The equilibrium point $E_{11}=(a_1/b_1,0)$ always exists and is a saddle point under the condition (5).

(3) The positive equilibrium point $E_{12} = (x^*,y^*)$exists, where

$$x^*=\frac{cd}{e_1-d}\ \text{and}\ y^*=(a_1-b_1x^*)(c+x^*), \qquad (6)$$

and it is a locally asymptotically stable point if the following condition holds:

$$d>\frac{e_1(a_1-b_1c)}{a_1+b_1c}. \qquad (7)$$

Moreover, if the condition $d < e_1(a_1- b_1c)/(a_1+b_1c)$ holds, the solution to the subsystem (4) approaches to a stable limit cycle even if the system is not a Kolmogorov system.

$$\begin{cases}\dfrac{dx_2(t)}{dt}=x_2(t)(a_2-b_2x_2(t))-\dfrac{x_2(t)y(t)}{\alpha+x_2(t)+\beta y(t)},\\ \dfrac{dy(t)}{dt}=y(t)\Big(-d+\dfrac{e_2x_2(t)}{\alpha+x_2(t)+\beta y(t)}\Big).\end{cases} \qquad (8)$$

The subsystem (8) is a Kolmogorov system if the following conditions are satisfied:

$$0<\frac{d\alpha}{e_2-d}<\frac{a_2}{b_2}. \qquad (9)$$

Elementary calculation yields that there exist at most three nonnegative equilibrium points of subsystem (8). Moreover, the stability of such equilibrium points can be studies by applying the local stability analysis to the subsystem (8) as the previous case. Thus we summarize these results as follows:

(1) The equilibrium point $E_{20}=(0,0)$ always exists and is a saddle point.

(2) The equilibrium point $E_{21}=(a_2/b_2,0)$ always exists and is also a saddle point under the condition (9).

(3) The positive equilibrium point $E_{22} = (\dot{x}, \dot{y})$ exists, where

$$\dot{x}=\frac{(a_2\beta+d-e_2)+\sqrt{(\beta a_2+d-e_2)^2+4b_2\alpha d}}{2b_2\beta}$$

and $$\dot{y}=\frac{(e_2-d)\dot{x}-d\alpha}{d\beta}, \qquad (10)$$

if the condition $0< \alpha d /e_2 - d < \dot{x} <a_2/b_2$ holds. In [14], the authors have investigated the local stability of the equilibrium point E_{22}.

Theorem 2.3 *[14] The positive equilibrium point $E_{22} = (\dot{x},\dot{y})$ if the Kolmogorov system (1) is locally asymptotically stable if one of the following sets of conditions is satisfied*

(1) $e_2\beta \ge 1$,

(2) $e_2\beta < 1$ and $\Delta_1^2 - 4\Delta_2 \le 0$,

(3) $e_2\beta < 1$ and $\Delta_1^2 - 4\Delta_2 > 0$, with $0< \dot{x} \le R_1$ or $R_2\le \dot{x} <1$. However, the solution of subsystem (8) approaches to a stable limit cycle for $R_1 < \dot{x} < R_2$. Here

$\Delta_1= d(1-\beta e_2)(d-e_2)/(\beta b_2e_2^2)$,

$\Delta_2 = \alpha d^2 (1-e_2\beta)/(b_2e_2^2\beta)$,

$R_1=\frac{1}{2}(-\Delta_1-\sqrt{\Delta_1^2-4\Delta_2})$ and

$R=\frac{1}{2}(-\Delta_1+\sqrt{\Delta_1^2-4\Delta_2})$

The term persistence is given to systems in which strictly solutions do not approach the boundary of the non-negative cones as time goes to infinity. Therefore, for the continuous biological system, survival of all interacting species and the persistence are equivalent.

Theorem 2.4 *Suppose that system (1) has no non-trivial periodic solutions in the boundary planes and satisfies the hypothesis of Theorem 2.3 and the condition $e_1(a_1-b_1c)/(a_1+b_1c) < d < a_1e_1/(a_1+b_1c)\}$ holds. Then the necessary conditions for the persistence of system (1) are*

$$\lambda_1=a_1-\frac{\dot{y}}{c}\ge 0, \qquad (11)$$

$$\lambda_2^*=a_2-\frac{y^*}{\alpha+\beta y^*}\ge 0, \qquad (12)$$

and the sufficient conditions for the persistence of system (1) are

$$\lambda_1=a_1-\frac{\dot{y}}{c}>0,$$

(13)

$$\lambda_2^* = a_2 - \frac{y^*}{\alpha + \beta y^*} > 0, \quad (14)$$

Proof. Note that the boundedness of system (1) is shown in

Thoerem 2.1 and E_{12} is locally stable under Kolmogrov condition (9). Since E_{12} and E_{22} are locally stable by the assumptions, the signs of the eigenvalues λ_2*and λ_1 determine the stability of the equilibrium points $E_4 = (x^*,0,y^*)$ and $E_5 = (0, \dot{x},\dot{y})$. In fact, if there are no non-trivial periodic solutions in the x2y plane and the equation (11) dose not hold(i.e. $\lambda_1 < 0$). Then there is an orbit in the positive cone, which approaches to E5. Hence, the condition (11) is one of the necessary conditions for the persistence. Similarly, we obtain the other necessary condition (12) for the persistence of system (1) by applying the same method as mentioned above to the equilibrium point E_4.

Now, we will use the abstract theorem of Freedman and Waltman [7] to figure out sufficient conditions for the persistence of system (1). In order to do this, consider the growth functions f_1, f_2 and f_3, where

$$\begin{cases} f_1(x,y,z) = a_1 - b_1 x_1(t) - \dfrac{y(t)}{c + x_1(t)}, \\ f_2(x,y,z) = a_2 - b_2 x_2(t) - \dfrac{y(t)}{\alpha + x_2(t) + \beta y(t)}, \\ f_3(x,y,z) = -d + \dfrac{e_1 x_1(t)}{c + x_1(t)} + \dfrac{e_2 x_2(t)}{\alpha + x_2(t) + \beta y(t)}. \end{cases} \quad (15)$$

Then it is shown that the following four conditions are satisfied:

(1) Clearly, we have $\partial f_i/\partial y < 0$, $\partial f_3/\partial x_i > 0$, i=1,2.

(2) Each prey population grows up to its carrying capacity in the absence of predators, that is, $f_1(0,0,0) = a > 0$ and $f_2(0,0,0) = a > 0$ and $\partial f_i/\partial x_i(x_1,x_2,0) = b_i < 0$(i=1,2) and $f_1(a_1/b_1,0,0) = 0 = f_2(0,a_2/b_2,0)$. Furthermore, the predator population dies out in the absence of preys, that is, $f_3(0,0,0) = -d < 0$.

(3) $\partial f_1/\partial x_2 = 0$ and $\partial f_2/\partial x_1 = 0$. There exists exactly one point $E_3 = (b_1/a_1,b_2/a_2,0)$ satisfying $f_i(b_1/a_1,b_2/a_2,0)=0$, i=1,2.

(4) In the absence of each prey species the predator can survive on the other prey. This is always true under the Kolmogrov conditions (5) and (9). There exists a unique $E_4=(x^*,0,y^*)$ and $E_5 = (0,\dot{x},\dot{y})$ satisfying $f_1(x^*,0,y^*) = f_3(x^*,0,y^*) = f_2(0,\dot{x},\dot{y}) = f_3(0,\dot{x},\dot{y})=0$. According to the Kolmogrov conditions (5) and (9), we can get that $f_3(a_1/b_1,0,0) > 0$ and $f_3(0,a_2/b_2,0) > 0$, respectively.

(5) It follows from (5),(9),(13) and (14) that the inequalities $f_3(a_1/b_1,a_2/b_2,0) > 0$, $f_1(0,\dot{x},\dot{y}) > 0$ and $f_2(x^*,0,y^*) > 0$ hold.

Therefore, by Freedman and Waltman theorem, system (1) persists under the hypotheses. ■

In order to substantiate our theoretical results, we display a numerical example by using Runge-Kutta method of order 4 for system (1) with initial value (1,1,1).

To do this we fix the parameters in system (1) as follows:

$a_1 = 1.5$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, $c = 1$, $d = 0.44$, $e_1 = 0.8$, $e_2 = 1$,

$\alpha = 0.5$ and $\beta = 0.1$.

Then system (1) satisfies the condition of Theorem 2.4. There system (1) is persistent, which means that all species survive forever when time goes up. In fact Figure 1 show such phenomena.

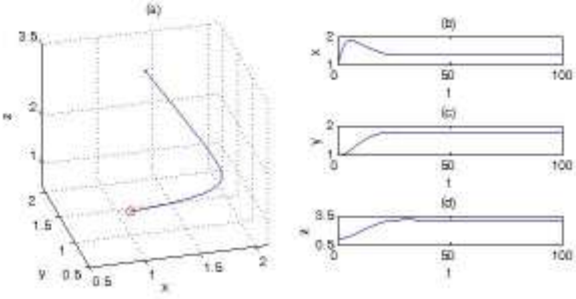

Figure 1 (a) A phase portrait of system (1) with an initial condition (1.0,1.0,1.0) and (b)-(d) time series for $x_1(t)$, $x_2(t)$ and y(t), respectively, when $a_1 = 1.5$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, $c = 1$, $d = 0.44$, $e_1 = 0.8$, $e_2 = 1$, $\alpha = 0.5$ and $\beta = 0.1$.

III. Conclusion and Discussion

In this paper, we have investigated a hybrid ecological system with two preys and one predator. Especially, we adopted Holling type II and Beddington-DeAngelis type functional responses. We have shown that this system is dissipative and persists under some conditions. And we have found out a suitable condition to survive the predator species.

Now, in order to observe the dynamic complexities of system (1), numerically, we fix the parameters except d as follows:

$a_1 = 1.5$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$, $c = 1$, $e_1 = 0.8$, $e_2 = 1$, $\alpha = 0.5$ and $\beta = 0.1$.

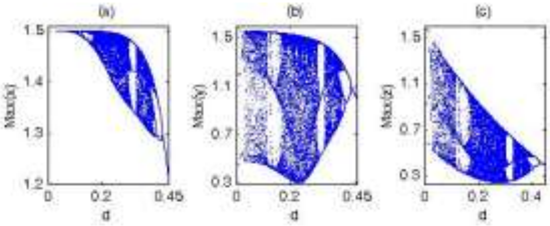

Figure 2 Bifurcation diagrams of system (1) with respect to d : (a), (b) maxima for the preys $x_1(t)$ and $x_2(t)$, respectively ; (c) maxima for the predator y(t).

Bifurcation diagrams of system (1) with respect to d is exhibited in Figure 2 when $0 \leq d \leq 4.5$. From Figure 2, we can figure out that system (1) undergoes various dynamical aspects such as periodic doubling bifurcations, periodic windows, chaotic regions and so on.

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