

# Submarine Optimal Depth Control applying Parseval's Theorem

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**Abstract**— In this paper a transfer function of submarine and steering gear are defined. Transfer functions have been analyzed and mathematical model of the system is presented. Optimal controller and its parameters are found using simulation of the whole system by Matlab and Simulink. Optimal parameters have been confirmed applying Parseval's theorem. Different types of controllers: P, PI and PID are analyzed. Responses for the whole system to the unity step function are given. To determine the stability of the system, analytical and graphical methods were used. A nonlinear system was also considered with two nonlinear elements to protect a steering gear from the huge signals. Further research for improvement of the model, including a certain disturbance is suggested.

**Keywords**—Submarine, optimal depth control, Parseval's theorem, Simulink, nonlinear system.

## I. Introduction

There are not many papers dealing with the submarine depth control systems, mainly because most submarines are used for navy and military purposes, i.e. classified. For analysing submarine motion is necessary to determine hydrodynamic forces and moments acting on the submarine movement. On the other hand, finding an appropriate mathematical model, from the existing literature, to describe depth-dynamic model of the submarine is also a quite challenging task. Once a suitable transfer function for the submarine and its steering gear were found, the analytical methods were used for analysing the system, as well as for designing suitable (optimal) parameters of the controller. Several non-linear characteristics also have to be included in the model using Simulink.

## II. Mathematical Model

### A. Dynamical Model

For analysing the submarine motion, it is necessary to determine hydrodynamic forces and moments acting on the submarine moving. Naturally, it must be done some simplifications on the model because the submarine is so complex system for control, with six degrees of freedom.

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Fig.1. shows submarine dynamic reference frame [5] in the earth-fixed, as well as body-fixed reference frame.

The kinetic equation are defined in terms of the body-fixed velocities: surge (u), sway (v), heave (w).

Define: yaw rate (r), pitch rate (q) and roll rate (p), where:

$$\dot{\psi} = r \quad \dot{\theta} = q \quad \dot{\phi} = p$$

and: yaw angle ( $\psi$ ), pitch angle ( $\theta$ ) and roll angle ( $\phi$ ).

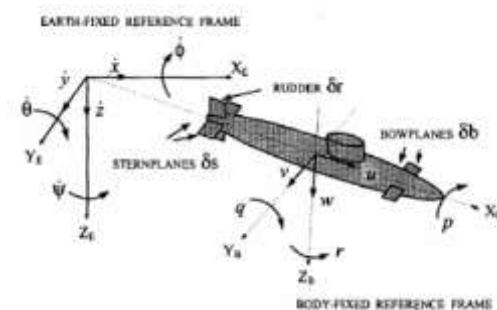


Fig.1. Submarine dynamical reference frame.

### B. Submarine Transfer Functions

Based on hydrodynamic coefficients, including forces and moments that acting on the submarine body, it is possible through the experiment of the model [6], [7] obtained an appropriate linearized transfer functions:

$$\frac{\psi(s)}{\delta_r(s)} = \frac{\left(\frac{2N_{uu}\delta_r}{N_{uv}}\right)}{\left(\frac{I_{zz}-N_z}{N_{uv}V^2}\right)s^2 + \left(\frac{-mX_g-N_{ur}}{N_{uv}V^2}\right)Vs + 1} \quad (1)$$

$$\frac{\theta(s)}{\delta_s(s)} = \frac{\left(\frac{2M_{uu}\delta_s V^2}{Z_g W - M_{uw} V^2}\right)}{\left(\frac{I_{yy}-M_j}{Z_g W - M_{uw} V^2}\right)s^2 + \left(\frac{mX_g - M_{uq}}{Z_g W - M_{uw} V^2}\right)Vs + 1} \quad (2)$$

$$\frac{\phi(s)}{\delta_a(s)} = \frac{\left(\frac{4K_{uu}\delta_a}{Z_g W}\right)}{\left(\frac{I_{xx}-K_p}{Z_g W}\right)s^2 + 1} \quad (3)$$

where:

- $(\delta_r, \delta_s, \delta_a)$  are the rudder, stern plane, aileron angles,
- $N_{uu\delta r}, M_{uu\delta s}, K_{uu\delta a}$  rudder, stern plane, aileron effectiveness,
- $N_{ur}, M_{uw}$  (body moment),
- $N_{\dot{r}}, M_{\dot{q}}, K_{\dot{p}}$  (added mass),
- $N_{ur}, M_{uq}$  (added mass cross term) are hydrodynamic coefficients,
- $X_g, Z_g$  are the coordinates of the CG relative to the centre of buoyancy.

Process of linearisation is assuming that changes of attitude, measured in Euler angles for roll, pitch and yaw are small, and with ignoring small quantities in the equations.

In this paper we are focused on the transfer function (2), which gives a relationship in a frequency (Laplace) domain between pitch angle over a stern plane. This transfer function is the second order (quadratic lag), with numerical estimated parameters [6], [7] and with chosen speed,  $V=[u,v,w]^T$ .

Dynamics for yaw and pitch depend on speed, whereas the roll dynamics are insensitive to speed. As mentioned before, our focus is on the pitch dynamics, which depend on a lateral speed:  $v=5\text{m/s}$ , which gives us the transfer function:

$$\frac{\theta(s)}{\delta_s(s)} = \frac{0.5}{s^2 + 0.4s + 1.11} \quad (4)$$

### C. Steering Gear Transfer Function

Having submarine by itself, it is meaningless. It has to be used with a steering gear (as an actuator). To control the submarine, it is necessary to have a larger signal than signal from the controller, only. The transfer function for the steering gear (S.G.) has been found as a first order transfer function:

$$S.G. = \frac{1}{0.5s + 1} \quad (5)$$

## III. Analysing System

### A. Closed Loop System

By combining the submarine and steering gear transfer functions and put them into the closed loop, we obtained the response, as on Fig.2.

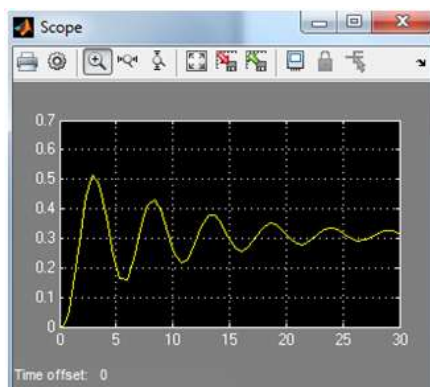


Fig.2. Closed loop response

From Fig.2., it is evident that system has a very slow response (rise time, and especially a settling time), as well as too big overshoot. Steady-state error is very big, too. Obviously, this response has to be improved by a right selection of the controller and its parameters.

Root locus of the combining steering gear and submarine is shown on Fig.3.

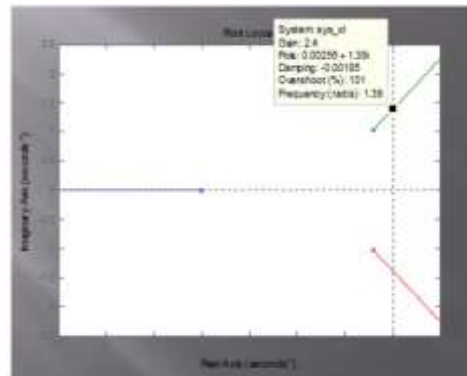


Fig.3. Root Locus

### B. Controller-Steering Gear-Submarine

The mathematical model of the closed loop system is shown on Fig.4. with a PID (Proportional-Integral-Derivative) controller.

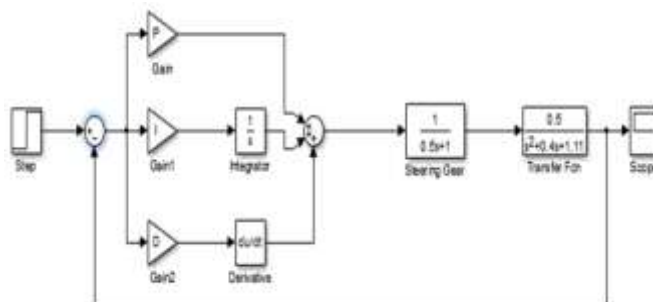


Fig.4. Mathematical model of submarine control system.

## IV. Designing Controller

First step for designing controller is selection of the controller type or a controller mode. Then, a fine tuning, as well as finding controller's optimal parameters.

### A. Controller Selection (general rules)

There are some well accepted guidance regarding the selection of the controller type to obtain a desired response [1], [3], [4]:

- Obtain an open loop response and determine what needs to be improved (it has been already mentioned in the previous sections, regarding

overshoot, steady state error, rise and settling time).

- Add a proportional control to improve the speed of the system response (particularly a rise time).
- Add a derivative control to improve the overshoot and the transient response.
- Add an integral control to eliminate the steady state error.
- Adjust each of those controller's parameters until obtain a desired overall response.
- And last, but not the least: make a controller as simple as possible.

The most likely effect of each of the controller parameters:  $K_p$ ,  $K_i$  and  $K_d$  (proportional, integral and derivative gain constants, respectively), on the closed loop system response, can be tabulated, as in the table below:

Table 1.

CL RESPONSE	RISE TIME	OVERSHOOT	SETTLING TIME	S-S ERROR
$K_p$ Increase	Decrease	Increase	Small Change	Decrease
$K_i$ Increase	Decrease	Increase	Increase	Eliminate
$K_d$ Increase	Small Change	Decrease	Decrease	Small Change

Note: Those correlations may not be exactly accurate, because  $K_p$ ,  $K_i$  and  $K_d$  are dependant on each other. In fact, changing one of those parameters can change the effect of the other two. For that reason, the table should be use as a reference or a guidance, only.

Referring to the Fig. 4. it can be seen that PID controller has to be involved in the system. PD mainly for obtaining less or no overshoot (if possible), much faster response, as well as to eliminate steady state error.

In many papers [2], dealing with marine vehicle control systems, there are recommendations for using PD controller, if the steady state error is not a dominant criteria, or if you already have one integrator in the system's transfer function, i.e., if the system is "type-1". However, in our model integral component is essential, because the submarine has a big steady state error. It can be reduced by bigger P component, but not eliminated. On the other hand, an "I" controller component will slow down the system dynamics, cause more oscillations and even could destabilise the system [1]. The right balance between all of those characteristics for an optimal control is required.

### B. Initial Controller's Settings

Having in mind previously mentioned, it is necessary to establish initial settings of PID controller, run the system with those parameters, and then apply a fine tuning for further improvement.

Starting point for this initial settings could be Ultimate Sensitivity tuning method or so called Ziegler-Nichols Second tuning method. Applying this, a closed loop tuning method is to leave P controller, only (i.e. disable integral and derivative gains).

Then, start up the process with the  $K_p$  at "low level" and gradually increase gain settings until the system starts to oscillate (i.e. having a sustain oscillations). At that point record  $K_p$ , which is the ultimate gain or critical gain,  $G_u$ .

That critical gain can be calculated and confirmed by using a numerical Routh's stability criteria or with a grapho-analytical method, i.e. with a root locus (see Fig.3.).

Based on period of the sustain oscillations,  $P_u$  and given  $G_u$ , determine the controller settings for PID, PI or P controller from the table below:

Table 2.

PID	PI	P
$K_p = 0.65G_u$	$K_p = 0.45G_u$	$K_p = 0.5G_u$
$T_I = \frac{P_u}{2}$	$T_I = \frac{P_u}{1.2}$	
$T_D = \frac{P_u}{8}$		

Note: Ziegler-Nichols Second tuning method is based on empirical formula and it is not so accurate. That means, calculating controller's parameters does not lead us to an optimal system, and rather gives us a range of the controller's parameters for a fine tuning.

In our example where:  $G_u=2.4$  and  $P_u=4.5$  sec, the ranges of the PID parameters are:  $K_p=1.56$ ,  $K_i=0.69$  and  $K_d 0.887$ . Those ranges will be used for determining the optimal parameters by using Parseval's theorem.

## v. Optimal Control by Parseval's Theorem

### A. Parseval's Theorem (basic concept)

Parseval's theorem is also known as Rayleigh's energy theorem, "connects" a time domain with a frequency domain, in a general form:

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |E(2\pi f)|^2 df \quad (6)$$

where:  $E(\omega) = F_{\omega}\{e(t)\}$  represents the continuous Fourier transform of  $e(t)$ , and  $\omega = 2\pi f$  is frequency in radians per second.

The LHS side is energy in time space while the RHS is energy in frequency (spectral) space.

In mathematics, Parseval's theorem usually refers to the results that the Fourier transform is unitary and that the sum (or integral) of the square of the function is equal to the sum of the square of its transform. Using Parseval's theorem it is possible to calculate integral from the LHS through the frequency spectrum (positive and negative) in "s" or Laplace's domain, using formulae:

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(s)E(-s)ds \quad (7)$$

$$E(s) = \frac{a(s)}{b(s)} \quad (8)$$

Combining those two formulae gives:

$$I_n = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{a(s)a(-s)}{b(s)b(-s)} ds \quad (9)$$

where: “n” is an order of the system, and “a” and “b” are the coefficients of the polynomes in the descending order of “s” :

$$a(s) = a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_0$$

$$b(s) = b_n s^n + b_{n-1}s^{n-1} + \dots + b_0$$

Integral (9) , for the fourth order system may be tranformed as a “table integral” ( $I_4$ ) in the form:

$$I_4 = \frac{1}{\Delta_4} [a_3^2(-b_0^2b_3 + b_0b_1b_2) + (a_2^2 - 2a_1a_3)b_0b_1b_4 + (a_1^2 - 2a_0a_2)b_0b_3b_4 + a_0^2(-b_1b_4^2 + b_2b_3b_4)]$$

$$\Delta_4 = 2b_0b_4(-b_0b_3^2 - b_1^2b_4 + b_1b_2b_3) \quad (10)$$

### B. Implementing Parseval’s Theorem

At present, it is common opinion that minimum of propulsion losses caused by submarine’s control should be adapted as a criterion of autopilot (controller) efficiency. Based on the losses through a deviation (dynamic error) from the desired depth, it can be define so called “Cost Function-C.F.”

$$C.F. = \int_{-\infty}^{\infty} |e(t)|^2 dt \rightarrow \min. \quad (11)$$

Cost Function can be calculated by using Parseval’s theorem, as well as through simulation to confirm the results of that integration.

A global minimum of the integration for defined range of controller’s parameter, from the previous section, will lead to the optimal submarine depth control system.

### C. Calculating Deviation in “s” domain

The deviation of the submarine from the desired depth can be calculated from Fig. 4. in the frequency domain:

$$E(s) = \frac{0.5s^3 + 1.2s^2 + 0.95s + 1.1}{0.5s^4 + 1.2s^3 + (0.95 + 0.5Kd)s^2 + (1.1 + 0.5Kp)s + 0.5Ki} \quad (12)$$

Taking the whole range of the controller’s parameters obtained by Ziegler-Nichols Second tuning method, and putting integral gain constant, Ki unchanged, it can be calculated Cost Functions (using Java programme language) for the whole spectrum of the Kp and Kd, which is tabulated in the Table 3.

Table 3.

	KD = 0.2	KD = 0.3	KD = 0.4	KD = 0.5	KD = 0.6	KD = 0.7	KD = 0.8	KD = 0.887
KP = 0.2	2.11895	2.03116	1.97164	1.92862	1.89607	1.87059	1.85011	1.83529
KP = 0.3	2.04192	1.95584	1.89798	1.85642	1.82512	1.80070	1.78112	1.76698
KP = 0.4	1.98223	1.89497	1.83696	1.79561	1.76464	1.74059	1.72135	1.70751
KP = 0.5	1.93818	1.84689	1.78702	1.74473	1.71327	1.68895	1.66959	1.65569
KP = 0.6	1.90883	1.81049	1.74702	1.70268	1.66994	1.64478	1.62484	1.61058
KP = 0.7	1.89392	1.78508	1.71617	1.66962	1.63383	1.60728	1.58634	1.57143
KP = 0.8	1.89307	1.77039	1.69395	1.64197	1.60433	1.57582	1.55347	1.53762
KP = 0.9	1.90892	1.76655	1.68013	1.62234	1.58099	1.54992	1.52573	1.50666
KP = 1.0	1.94432	1.77413	1.67472	1.60953	1.56349	1.52924	1.50276	1.48418
KP = 1.1	2.00077	1.79421	1.67800	1.60349	1.55166	1.51352	1.48427	1.46387
KP = 1.2	2.08516	1.82855	1.69056	1.60438	1.54545	1.50260	1.47005	1.44749
KP = 1.3	2.20700	1.87981	1.71335	1.61253	1.54493	1.49644	1.45997	1.43488
KP = 1.4	2.38215	1.95203	1.74780	1.62851	1.55030	1.49007	1.45398	1.42593
KP = 1.5	2.63883	2.05135	1.79598	1.65317	1.56193	1.49860	1.45208	1.42060
KP = 1.56	2.85278	2.12772	1.83268	1.67260	1.57212	1.50317	1.45293	1.41914

The minimum value of the calculated Cost Function by using Parseval’s theorem, gives the optimal controller's parameters:

$$Kp=1.56, Ki=0.69 \text{ and } Kd=0.877.$$

It can be noticed that those results confirm Ziegler-Nichols choice for the optimal parameters.

However, with a small tuning of those parameters, based on Table 1. it is possible to get slightly better response of the system by changing Kp=1.3, while two others remain the same.

## VI. Simulation by Matlab and Simulink

### A. Simulation of the Linear System

By using Matlab and Simulink, it is possible to confirm those optimal parameters in time domain. Of course, at the same time, it proves a correctness of the Parseval’s theorem, in the case of Cost Function, as an integral of quadratic dynamic error.

Simulation of the whole system: controller-steering gear-submarine is shown on Fig.5. as well as, with some additional blocks, including two nonlinearities, which will be discuss later.

Scope1 (on Fig.6.1., Fig.6.2. and Fig.6.3., in Appendix) shows integral of quadratic dynamic error for two randomly selected controller’s parameters (from already defined range), and for optimal parameters, respectively. Optimal parameters give the smaller value of the integral than another two.

### B. Simulation of Nonlinear System

In the case of a huge “derivative kick”, where Kd is relatively big, as well as a dynamic error (especially at t=0), it is advisable to implement nonlinear element (saturation blocks in Fig.5.). Those blocks have a positive effect on stability, if chosen properly.

The first saturation block is used as a stern plane limiter. This helps to limit the movement (position) of stern plane and to protect the steering gear from damage.

The second saturation block acts as a stern plane rate limiter and protects the stern plane from the very fast movement.

Note: to include the stern rate limiter in the system, it is necessary to modify steering gear block from Fig.4.

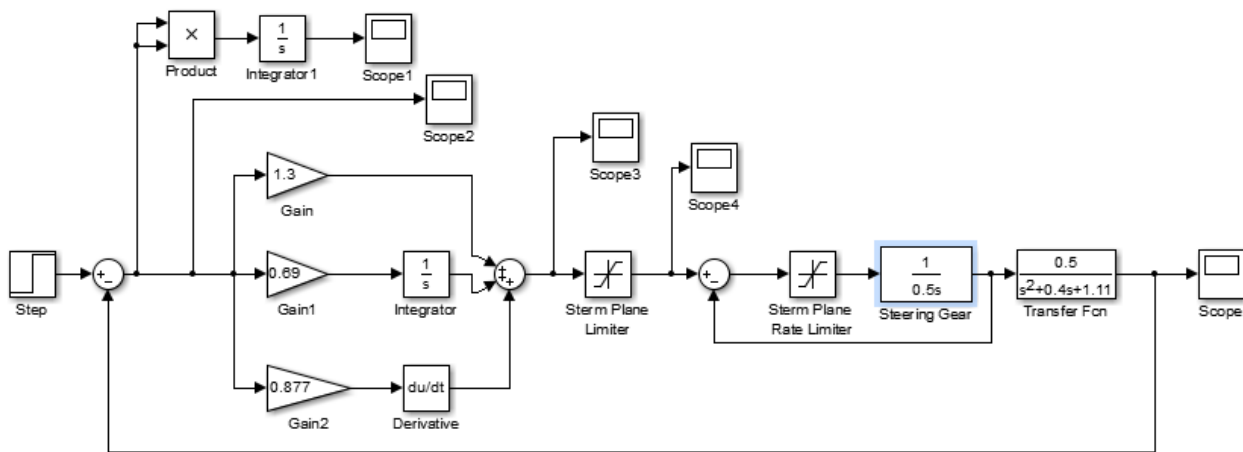


Fig.5. Simulation model.

Several different limits (from 0.5-5) have been applied in both saturation blocks. Fig.7. (in Appendix) shows a slightly bigger integral of quadratic dynamic error if those blocks are included in the system, but still better than in the cases of randomly selected parameters.

Anyway, those blocks can be used as a precaution for the steering gear.

If nonlinearities were introduced in the system, then Parseval's theorem cannot be applied, but a simulation, only.

## VII. Conclusion

Mathematical model of the submarine depth control system is defined. Different controller's types have been examined, then PID controller selected. Ziegler-Nichols tuning method is used to find an initial range of the controller's parameters. Criteria for optimal controller's parameters is defined through the Cost Function. Parseval's theorem for calculating those parameters is used. Some small tuning is performed. Matlab and Simulink are used to simulate the whole system and to prove a correctness of the Parseval's theorem. Two nonlinearities (stern plane and rate stern plane limiters) are introduced. Further research could lead to introducing a disturbance to the system (such as a sea current) and find how it effects the system's performance.

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## Appendix

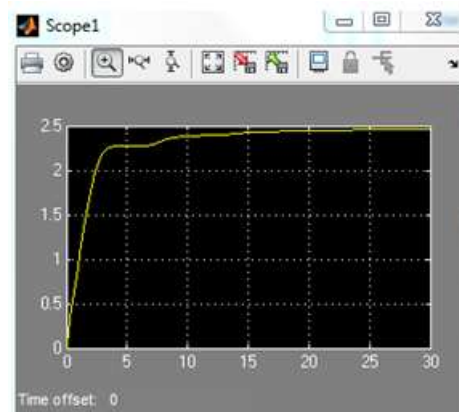


Fig.6.1.  $K_p=0.1$ ,  $K_i=0.69$ ,  $K_d=0.1$

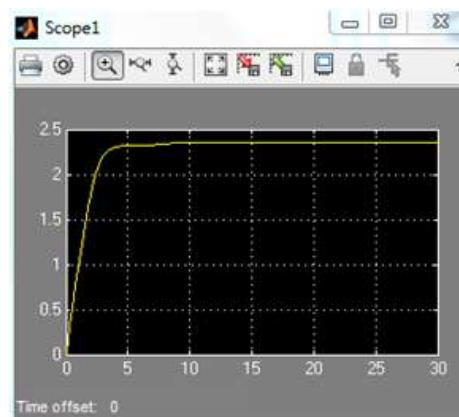


Fig.6.2.  $K_p=0.5$ ,  $K_i=0.69$ ,  $K_d=0.5$

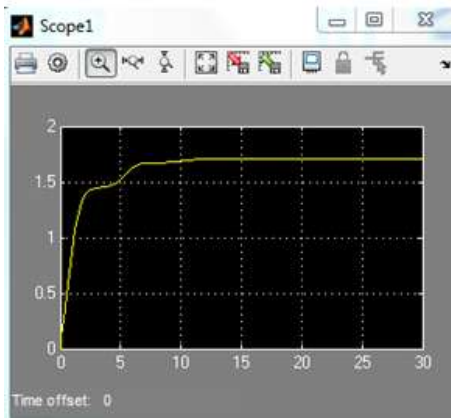


Fig.6.3.  $K_p=1.3$ ,  $K_i=0.69$ ,  $K_d=0.877$



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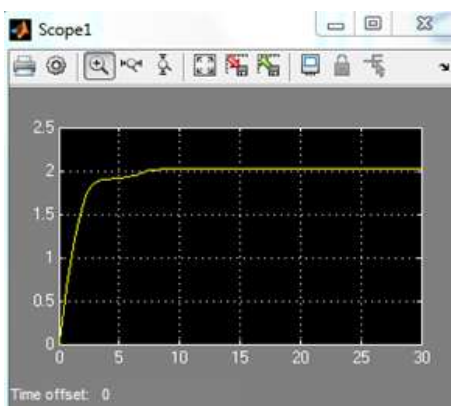


Fig.7. With two saturation blocks:  
 $K_p=1.3$ ,  $K_i=0.69$ ,  $K_d=0.877$



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