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# Multiple Model Adaptive Control Design

## A Computer Algebra Approach

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*Abstract*—**The multiple model adaptive control (MMAC) method potentially has the ability of providing excellent control performance over a wide range of parameter variations. A key part in the design of a MMAC system is the choice of candidate models. In this work, a method to optimally determine these candidate models is proposed. The method exploits the relationship between Grobner bases and polynomial spectral factorization through the notion of sum of roots. Using computer algebra techniques, the symbolic solution to the Algebraic Riccati Equation, and hence the $H_2$ optimal control problem, can be obtained. The symbolic solution gives the relationship between the optimal controller gains and the uncertain parameters. Hence, the candidate controllers in the MMAC system could be selected based on this relationship to best counter the parameter variations. To illustrate the effectiveness of the proposed method, a case study of a MMAC implementation on a quarter car active suspension with varying sprung mass is presented. The candidate controllers chosen using the proposed method was compared to a similar MMAC system where the candidate controllers were chosen based on equal sprung mass spacing. Simulations were performed under varying sprung masses, and using changing road profiles as a disturbance input. The performance criteria were vertical sprung mass acceleration, tyre force and suspension deflection. The results demonstrated the advantages of the proposed method where improved performances were obtained for all three criteria.**

*Keywords*—**MMAC, sum of roots, Grobner bases, spectral factorization, active suspension**

## I. Introduction

Uncertainty exists in systems due to unmodelled dynamics, unknown system parameters, disturbances, and/or process changes. An effective control system needs to be capable of preserving stable and robust system performance in spite of the inherent uncertainties. When uncertainties in the system are relatively small, modern linear time invariant (LTI) control system theories, such as $H_\infty$ and $\mu$ synthesis, could sufficiently guarantee closed-loop performance under certain specified conditions.

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Multiple model adaptive control (MMAC) is a control concept that extends well established linear control system design methods to be applicable over a wider region of uncertainty. When the actual parameter uncertainty is significantly extensive, the range of parameter values can be divided into sets of smaller ranges. Every set is then associated with a distinct plant model. Hence, within each set, parameter uncertainties from the designated plant model is significantly reduced. Thus, it is possible to design dynamic compensators for each plant model such that satisfactory performance is always obtained within each set.

The basic idea behind MMAC is to have a set of candidate models enveloping the entire operating region of the plant. For each candidate model, a controller that produces the required plant performance, for that particular operating condition, is determined a priori. During system operation, using some probabilistic function, the degree of matching between each candidate model and the plant is determined. Weights are determined for each candidate controller based on the probabilistic function. A close match would be assigned a weight close to 1, and a bad match would have a weight close to 0. The control input into the plant is then the weighted sum of all candidate controllers. The plant dynamics would always be monitored, and the probabilistic function value and control input would be continuously updated accordingly [1,2,3].

A key part to the successful implementation of a MMAC system is the choice of candidate models. Thus far, this is very much a heuristic process. In this work, we propose a method to optimally determine these candidate models. The proposed method makes use of computer algebra techniques to formulate the analytical solution to the optimal control problem in terms of symbolic parameters. This provides a notion on how changes in the uncertain parameters affect system performance. Consequently, the candidate models can be chosen such that the effect of variation of parameters can be reduced in the most effective manner.

## II. *H₂* Optimal Control

In this study, for each candidate model, a $H_2$ optimal controller is designed. Consider the control problem as illustrated in Fig. 1. Here, $P$ represents the plant to be controlled, while the fixed feedback gain is denoted by $F$.

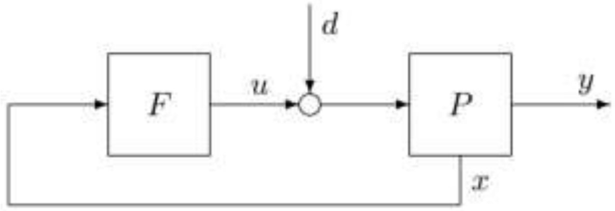


Figure 1 State feedback design

The signals $x, u, y$ and $d$ are respectively, the plant state, the control input, the plant output and disturbance input. Essentially, the problem is to reduce the impact of the disturbance, $d$, on the output, $y$ subject to constraints on the control input $u$ caused by actuator limitations. Hence, the goal is to ensure stability of the closed-loop system, while minimizing the $H_2$-norm of the transfer function matrix from $d$ to $[y\, u]^T$. This may be achieved through the use of an optimal state feedback gain, $F$. The optimization can be performed through the quadratic cost function given by

$$E(P,F) = \int_0^\infty (\|y(t)\|_2^2 + \|u(t)\|_2^2)dt \qquad (1)$$

where $\|\cdot\|_2$ is the (Euclidean) 2-norm. Then, the greatest attainable efficiency is given by

$$E^*(P) = \inf_{F\ stabilizing} E(P,F) \qquad (2)$$

Based on the Algebraic Riccati Equation (ARE), a solution can be obtained as follows. The plant of degree $n$ with $m$ inputs and $p$ outputs is given a state-space model as

$$P = \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad (3)$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, with $C \in \mathbb{R}^{p\times n}$. Suppose that ($A$, $B$) and ($C$, $A$) are stabilizable and detectable. Thus, the following defines the best way to achieve the optimal controller.

Let $X = X^T \geq 0$ be the solution of the ARE, given by

$$A^T X + XA - XBB^T X + C^T C = 0 \qquad (4)$$

Then, the $H_2$ optimal state feedback gain is given by

$$F_{opt} = -B^T X \qquad (5)$$

The best obtainable control performance may be obtained from the cost function

$$E^*(P) = tr\{B^T XB\} \qquad (6)$$

The ARE may be written in the Hamiltonian matrix form, which is of degree $2n \times 2n$ and composed in the form of [4].

$$H = \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}. \qquad (7)$$

Then, spectral factorization of $H$ needs to be performed [5]. Assume that the eigenvalues in the open left half plane are $\lambda_i, i = 1,2,\dots,n$. By seeking a basis for $v$ wherein $v$ is the invariant subspace of $H$ according to the $i's$, one obtains $X_1$, $X_2$ such that

$$v = Range \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, X_1, X_2 \in C^{n\times n}. \qquad (8)$$

Then, the solution takes the form of

$$X = X_2 \times X_1^{-1} \in R^{n\times n} \qquad (9)$$

From the eigenvectors associated with the eigenvalues, the matrix $[X_1^T\ X_2^T\ ]$ is constructed in the open left half plane. Hence, the solution $X$ can be calculated via (9).

$X$ will vary according to the value of the plant parameters given by $A, B$ and $C$. If the solution to the ARE could be found analytically, in the algebraic form, the relationship between $X$, and hence, $F$, with the varying (uncertain) parameters can be established. Thus, the candidate models can be chosen based on this relationship to give the most optimal result. In order to obtain the algebraic analytical solution to the ARE, the notion of sum of roots (SoR) can be used.

# III. Algebraic Solution of the ARE

Although there exist a number of numerical methods to solve spectral factorization problems [6], their use is limited in that they cannot handle systems with symbolic parameters. It has recently been shown that the concept of SoR reveals an intriguing connection between polynomial spectral factorization and Gröbner Bases [7].

The approach that is used in this paper expresses the solution of the ARE in terms of the spectral factor coefficients as well as by means of the shape basis concept for the SoR. This solution comprises of a polynomial which associates the parameters to the SoR, and creates algebraic expressions for the elements $X$ in the form of polynomials of the SoR and rational functions in terms of the symbolic parameters. Parameters of the system and the $X$ matrix are consequently associated through an algebraic approach, and a direct analysis can be conducted on the impact of the parameters on the cost function, and subsequently, the controller gains. The approach is made up of the following steps:

1. Use polynomial spectral factorization to relate the parameters to the SoR from the Hamiltonian matrix $H$.
2. Obtain the eigenvectors of $H$ in symbolic form for the corresponding eigenvalues.
3. Construct $[X_1^T\ X_2^T]^T$ consisting of two matrices — the first comprising of polynomials in the form of coefficients of the spectral factor and the other comprising of monomials in the eigenvalues. This separates the coefficients of the spectral factor from the eigenvalues;
4. Obtain an expression of matrix $X$ relating to the coefficients of the spectral factor from the divided result; so, matrix $X$ could be expressed in the form of the Sum of Roots through the results from Step 1.

Consider the polynomial $f(\lambda)$, where $\lambda_i \neq \lambda_j$ for all $i, j;\ i \neq j$. Assume the SoR to be

$$\sigma = -(\lambda_1 + \lambda_2 + \dots + \lambda_n), \qquad (10)$$

and let

$$\begin{aligned} f(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n) \\ &= \lambda^n + \sigma\lambda^{n-1} + m_{n-2} + \cdots \\ &+ m_0. \end{aligned} \qquad (11)$$

While every element $x_{i,j} \in X$ is symmetric around the origin and is also a rational function of $\lambda_i$'s, it would be realized that each $x_{i,j}$ could also be shown to be a rational function of $\sigma$ and $m_j$, $j = 0,1,\ldots,n-2$. Furthermore, because of the fact that the characteristic polynomial of $H$ is equivalent to $(-1)^n f(-\lambda) f(\lambda)$, and along with the characteristics of the ideal of spectral factorization, the $m_j's$ can become an expression of polynomials in $\sigma$ only [8]. Consequently, $x_{i,j}'s$ tend to be an expression of rational functions in $\sigma$. Additionally, $\sigma$ can be taken as the largest real root of the polynomial. By comparing the coefficients of the two sides of

$$\det(\lambda I - H) = (-1)^n f(-\lambda) f(\lambda), \tag{12}$$

the set $\mathcal{G}$ can be obtained in terms of the polynomials in $\sigma$ as well as in $m_j's$, that gives a Gröbner basis $< \mathcal{G} >$, of the ideal of the spectral factorization. The ideal includes a shape basis, with $\sigma$ being the separating element that can effectively be calculated by means of the basis conversion (changing the order) method. This indicates that the characteristic polynomial $S_f(\sigma)$ of $\sigma$ can be obtained, and that any polynomial in $m_j's$ can be written in terms of $\sigma$ only.

As a result, the aim is to obtain an expression of $X$ in the form of $\sigma$ and $m_j's$. Recall that $\sigma$ and $m_j's$ are elementary symmetric polynomials (up to sign) in $\lambda_i$'s, hence, any symmetric polynomial in $\lambda_i$'s can be expressed as a polynomial in $\sigma$ and $m_j's$. Writing an eigenvalue of $H$ as $\lambda$, and by means of symbolic computation, one can find an eigenvector $v(\lambda) \in \mathbb{R}[\lambda]^{2n}$ such that

$$(\lambda I - H) v(\lambda) = 0 \tag{13}$$

where the greatest common divisor of all elements of $v(\lambda_i)$ is equal to one. As $(\lambda_i) \in \mathcal{V}$, one can express $[X_1{}^T\ X_2{}^T]^T$ in terms of $\lambda_i$'s using $v(\lambda_i)$. The task is then to rewrite it in terms of $\sigma$ and $m_j's$. Since $f(\lambda_i) = 0$, it is in fact sufficient to consider only the remainder on dividing each element of $v(\lambda)$ by $f(\lambda)$. One can immediately deduce that the remainder is in $L = K[\lambda]/f(\lambda)$, where $K = \mathbb{R}[\sigma, m_{n-2}, \ldots, m_0]/< \mathcal{G} >$. In other words, the remainder is a polynomial in $\lambda$ of degree up to $n-1$ whose coefficients are polynomials in $\sigma$ and $m_j's$. Therefore, $[X_1{}^T\ X_2{}^T]^T$ can be expressed as the transpose of a Vandermonde-like matrix, that is,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \left( v(\lambda_1)\, v(\lambda_2) \ldots v(\lambda_n) \right) = \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda^{n-1}{}_1 & \lambda^{n-1}{}_2 & \ldots & \lambda^{n-1}{}_n \end{bmatrix} = \begin{bmatrix} \tilde{X}_1 V^T \\ \tilde{X}_2 V^T \end{bmatrix} \tag{14}$$

where $V$ is the Vandermonde matrix and $\tilde{X}_1, \tilde{X}_2 \in K^{n\times n}$. Notice that variable sets $(\lambda_1, \ldots, \lambda_n)$ and $(\sigma, m_{n-2}, \ldots, m_0)$ are neatly separated, and one can effectively eliminate $\lambda_i$'s and obtain $X$ as

$$X = \tilde{X}_2 \tilde{X}_1{}^{-1} \tag{15}$$

where each element of $X$ is an element of $\frac{1}{\det(\tilde{X}_1)} \mathbb{R}[\sigma, m_{n-2}, \ldots, m_0]$. Finally, the unique positive definite solution can be obtained from the largest real root of $S_f(\sigma)$.

# IV. Case Study: A Quarter Car Suspension Model

The case study presented here follows the MMAC suspension system in [9]. Fig. 2 shows a schematic of the quarter car suspension system. Here, $m_1$ and $m_2$ are respectively the sprung and unsprung masses, and $k_t$ represents the tyre stiffness. For simplicity, the shock absorber damping and spring forces, as well as the controller actuator forces are lumped together, represented here as $u$.

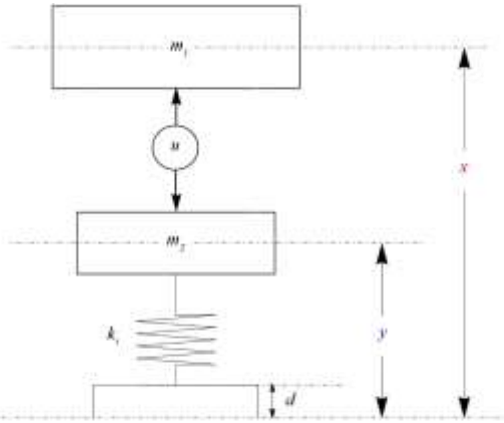

Figure 2 Simplified quarter-car model

The dynamic equations of the suspension can be written as

$$\begin{aligned} m_1 \ddot{x}(t) &= u(t) \\ m_2 \ddot{y}(t) &= -u(t) - k_t\big(-d(t) + y(t)\big) \end{aligned} \tag{16}$$

where $x, y$ and $d$ are respectively, the vertical displacements of the sprung mass, of the unsprung mass, and of the road profile.

Let

$$x(t) = [x(t) \quad y(t) \quad \dot{x}(y) \quad \dot{y}(t)]^T \tag{17}$$

be the state vector. Then, the state space representation is given by(Camino et al. 1999).

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \tag{18}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-k_2}{m_2} & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ -\frac{1}{m_2} \end{bmatrix}, \tag{19}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix}$$

In this case study, it is assumed that all parameters are constants, with the exception of $m_1$, which may take any value in the range of 0 to 500 kg. Using the proposed method, a relationship between the cost function $E^*(P)$ and $m_1$ was found. The result is a long and complex equation, which would be impractical for use without the availability of computer algebra software. This relationship is represented by the curve in Fig. 3.

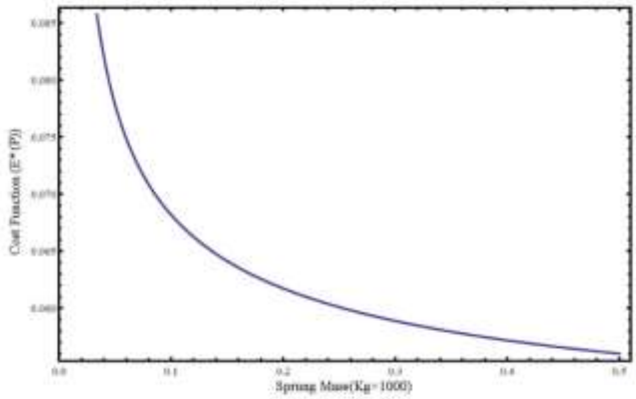

Figure 3 Cost vs sprung mass value

By dividing the cost into four equal intervals, four values of $m_1$, corresponding to the midpoint of each cost interval were selected to be used for the candidate models in the MMAC system. The corresponding $H_2$ optimal controller for each candidate model was subsequently determined. For brevity, we denote this as Method A.

For comparison, a second MMAC system was designed by dividing the range of $m_1$ into four equal intervals. The candidate models were then selected from the midpoint value of each interval. The $H_2$ optimal controller for each candidate models were also determined, and this was denoted as Method B.

The road profile was taken as the disturbance input. This was represented by a square wave of 20-second period and 2 cm amplitude. In performing the simulations, a piecewise varying scenario was adopted where, in addition to the varying road profile, the sprung mass value also changes at 5 second intervals, as shown in Fig. 4.

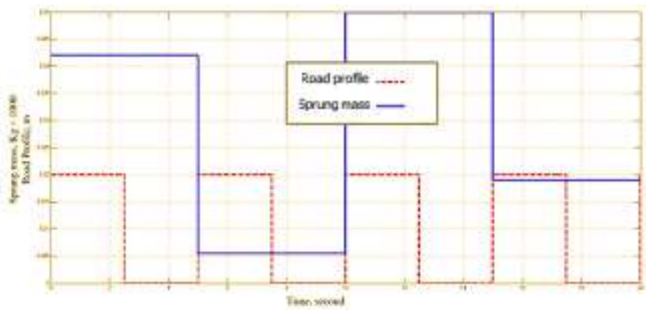


Figure 4 Scenario for the simulation tests

A Luenberger observer was used to estimate the operating plant dynamics and the $H_2$ regulator corresponding to each candidate model was designed. Table 1 gives the sprung mass values for each candidate model, and the corresponding regulator and estimator gains obtained Method A, and Table 2 give the values obtained using Method B.

Table 1 Regulator and estimator gains for Method A

| Candidate Model | Regulator Gain | Estimator Gain |
|---|---|---|
| A1(30.5kg) | [77.4597 -67.2658 3.64502 1.56807] | [119.509; 31.3615; 2539.66; -1345.32] |
| A2(83.5kg) | [77.4597 -77.1053 6.37309 0.302132] | [76.3245; 6.04264; 927.661; -1542.11] |
| A33(177kg) | [77.4597 -76.3903 9.24355 -0.523656] | [52.2235; -10.4731; 437.625; -1527.81] |
| A4(369kg) | [77.4597 -72.5249 13.0676 -1.11066] | [35.4135; -22.2131; 209.918; -1450.5] |

Table 2 Regulator and estimator gains for Method B

| Candidate Model | Regulator Gain | Estimator Gain |
|---|---|---|
| B1(71.25kg) | [77.4597 -76.4836 5.86761 0.500424] | [82.3525; 10.0085; 1087.15; -1529.67] |
| B2(193.75kg) | [77.4597 -76.0158 9.65165 -0.607725] | [49.815; -12.1545; 399.792; -1520.32] |
| B3(316.25kg) | [77.4597 -73.4449 12.1595 -1.00486] | [38.449; -20.0972; 244.932; -1468.9] |
| B4(438.75kg) | [77.4597 -71.469 14.165 -1.21941] | [32.285; -24.3883; 176.546; -1429.38] |

The time response simulation results for the sprung mass vertical acceleration, tyre deflection and suspension deflection are shown in Figs. 5, 6 and 7 respectively, and their RMS values are given in Table 4.4. From all three time responses, the improvement obtained from using Method A compared to Method B is most evident between the time of 5 – 10 seconds, where the oscillatory response is greatly reduced. It should be noted that the sprung mass value is smallest over this period. From Fig. 3, in the low sprung mass region, the cost function value changes significantly

| Criterion | RMS Values | | |
|---|---|---|---|
| | Method A | Method B | Percentage Improvement |
| Acceleration (mm/s$^2$) | 3.46 | 5.03 | 31.3 % |
| Tyre force (N) | 599 | 601 | 0.33 % |
| Suspension deflection (mm) | 0.050 | 0.053 | 5.67 % |

with small changes in sprung mass. The MMAC system designed using Method A had more candidate models concentrated in this region, compared to the higher sprung mass region, where the cost function is significantly less affected by changes in sprung mass values. Hence, the advantage of selecting candidate models based on the relationship between cost function and the varying parameter is evident.

The RMS values also show improvements gained from using Method A, particularly for sprung mass acceleration. The RMS values obtained here were taken over the whole simulation time. Outside the time of 5 – 10 seconds period, the differences between Method A and Method B are minimal. Hence the overall RMS values is lowered, particularly for the suspension deflection criterion.

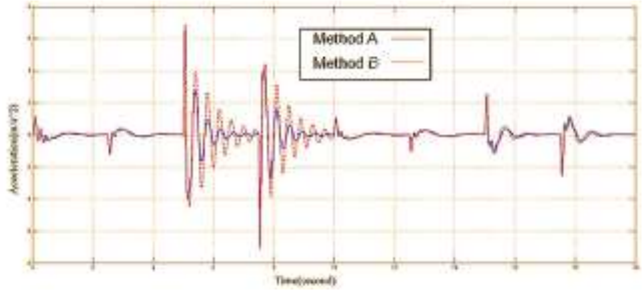


Figure 5 Sprung mass acceleration response

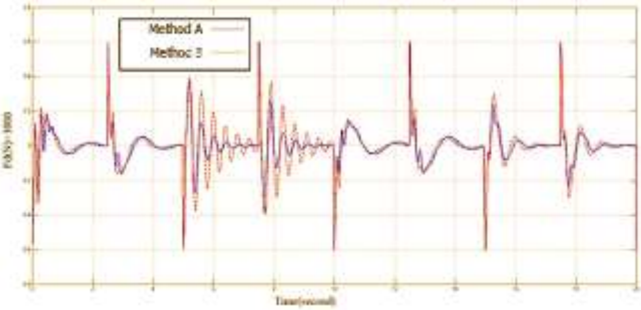


Figure 6 Tyre force response

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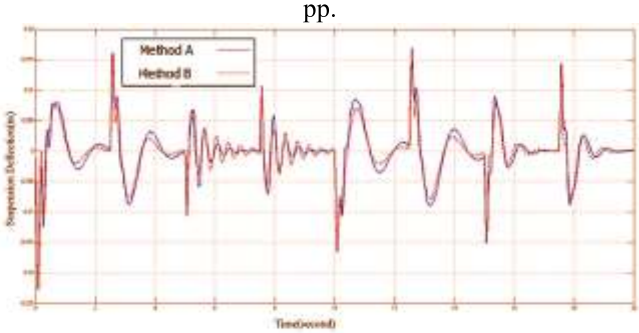


Figure 7 Suspension deflection response

Table 3 RMS values for the performance criteria

# V. Conclusion

The MMAC is a control method that has potential in wide areas of applications, including parameter varying and nonlinear systems. A method has been proposed to enable the candidate models to be selected effectively so as to provide improved performance. The availability of computer algebra software has enabled the relationship between the solution of the ARE and the varying parameter be obtained in the symbolic algebra form. From this relationship, the MMAC candidate models can be selected based on how changes in the varying parameter would affect the system performance. A test case study of MMAC design for a quarter car suspension with changing sprung mass values highlights the advantages of using the proposed method.

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