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Free Vibration Analysis of Cross-ply Laminated Composite Thick Plates

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Abstract*—*In this study, it is aimed to analyze the free vibration behavior of cross-ply laminated composite rectangular thick plates with mixed finite element method based on the Gâteaux differential. Mixed finite element model has eight unknowns, displacement, bending and twisting moments, rotations and shear forces. In the free vibration analysis, laminated composite rectangular thick plates are considered for different geometrical and material parameters, lamination scheme and boundary conditions. Accuracy of the presented functional and mixed finite element formulation is shown by comparing the results of numerical examples with the ones available in the literature. The results obtained in this study are found to be in good agreement.

Keywords*—*free vibration analysis, composite thick plates, Gâteaux differential, mixed finite element method

I. Introduction

Composite materials have high strength when compared to isotropic materials, easily shaped and light due to their specific weight. In order to understand the response of composite structural elements, lots of studies are conducted in literature. Studies in literature generally present approximate solution of the problem with Finite Element Method (FEM), Finite Difference Method, Rayleigh-Ritz Method and Galerkin Method instead of the exact solution. In this study, free vibration analysis of cross-ply laminated composite thick plates is carried out by mixed finite element method based on the Gâteaux differential approach. The Gâteaux differential is applied to elastic, viscoelastic and composite beams [1-4]. Detail information about Gâteaux differential can be found in the literature [5].

There are many studies about the free vibration analysis of laminated thick plates in literature and various analytical and numerical methods are considered for analysis. Dai et al. [6] used a mesh free method for static and free vibration analysis of shear deformable laminated composite plates. Yoshiki and Toshihiro [7] studied the analytical models for vibration of cross-ply laminated thick plates. Luccioni and Dong [8] used Levy-type finite element analysis of vibration and stability of thin and thick laminated composite rectangular plates. Khdeir and Reddy [9] studied free vibrations of laminated composite plates using second order shear deformation theory. Liu et al. [10] used the confirming radial point interpolation method for the static and free vibration analysis of laminated composite plates.

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Batra and Aimmanee [11] studied vibration of thick isotropic plates with higher order shear and normal deformable plate theories. Aagaah et al. [12] examined natural frequencies of laminated composite plates using third order shear deformation theory. Zhou et al. [13] used finite layer method for the free vibration analysis of thick layered rectangular plates with point supports. Shimpi and Ainapure [14] used layer-wise trigonometric shear deformation theory for free vibration analysis of two-layered cross-ply laminated plates.

In this research, a new functional is constructed through a systematic procedure based on the Gâteaux differential for the free vibration analysis of cross-ply laminated composite thick plates. Mixed finite element method is used for the derivation of element matrix by selecting the linear shape functions.

In order to verify the accuracy of the derived functional and presented mixed finite element formulation, numerical examples are solved. Results are compared with the ones available in the literature. It has been shown that, the presented mixed finite element formulation can accurately predict the vibration frequencies of the cross-ply laminated composite thick plates.

II. Method

A Cartesian coordinate system (x, y, and z) is defined on the central axis of the plate. Field equations of thick plates can be found in [15]. In this field equations, q is the uniform load which is applied at the top surface of the laminated plate. M_x, M_y, M_{xy}, Q_x and Q_y are the moment resultants and transverse force resultants of which their positive directions are illustrated in Figure 1. w, Ω_x are the geometric parameters denote the displacement of a point (z) in the laminated plate and the cross-sectional rotation about x axis.

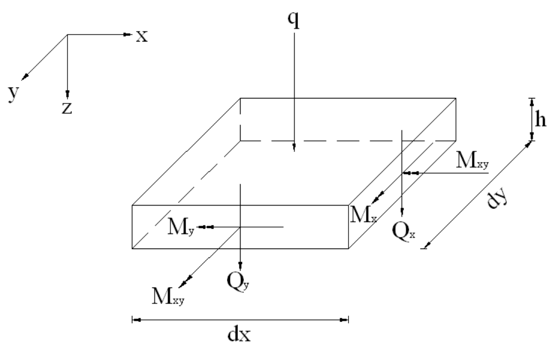

Figure 1. Internal Forces

The flexural rigidity matrix, D_{ij}, in the field equations are defined as:

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N}\left(\overline{Q}_{ij}^{(k)}\right).(z_k^3 - z_{k-1}^3). \qquad (\mathrm{i,j} = 1,2,6) \qquad (1)$$

z_{k+1} and z_k are the coordinates of the upper and lower surfaces of the k^{th} layer.

For orthotropic materials of each layer of laminated composite plates, the two-dimensional stress-strain equations for the k^{th} layer can be written under the assumption of plane stress as:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}^{(k)}. \qquad (2)$$

where σ_1, σ_2, τ_{12} are the stresses and ε_1, ε_2, γ_{12} are the linear strain components referred to the principal material coordinates of layer (1-2). $Q_{ij}^{(k)}$ are elastic constants of the k^{th} layer and can be written as follows:

$$\begin{aligned} Q_{11} &= \frac{E_1}{(1-\mu_{12}\mu_{21})}. \\ Q_{12} &= Q_{21} = \frac{\mu_{12}E_2}{(1-\mu_{12}\mu_{21})}. \\ Q_{22} &= \frac{E_2}{(1-\mu_{12}\mu_{21})}. \\ Q_{66} &= G_{12}. \end{aligned} \qquad (3)$$

with E_i being Young's modulus in the i^{th} material direction, μ_{ij} is the Poisson ratio and G_{ij}, is the shear modulus of the i-j plane.

The constitutive equations for the k^{th} orthotropic layer are transformed to the laminate coordinates (x, y, z) as follows:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^{(k)} = \left[\overline{Q}_{ij}\right]^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{21} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}^{(k)}. \qquad (4)$$

where $\overline{Q}_{ij}^{(k)}$ are the transformed elastic constants or stiffness matrix with respect to the laminate coordinates (x, y, z) [16].

The elements of $\overline{Q}_{ij}$ matrix can be written as follows:

$$\begin{aligned} \overline{Q}_{11} &= Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta. \\ \overline{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta). \\ \overline{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta c\mathrm{os}\theta. \\ \overline{Q}_{22} &= Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta. \\ \overline{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta. \\ \overline{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta). \end{aligned} \qquad (5)$$

where θ is the angle between the global axis and the local x-axis of each layer as shown in Figure 2.

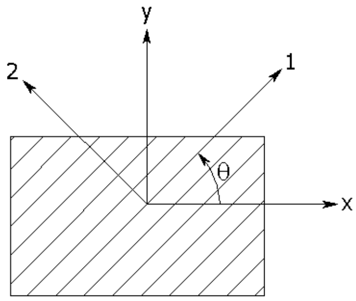

Figure 2. Global and local axis

The dynamic and geometric boundary conditions of composite plates can be written in symbolic form as follows:

Dynamic boundary conditions:

$$\begin{aligned} \mathrm{M} - \mathrm{M} &= 0. \\ \mathrm{Q} - \mathrm{Q} &= 0. \end{aligned} \qquad (6)$$

Geometric boundary conditions:

$$\begin{aligned} -\Omega + \Omega &= 0. \\ -\mathrm{w} + \mathrm{w} &= 0. \end{aligned} \qquad (7)$$

Equilibrium, constitutive and field equations can be written in operator form as

$$\begin{aligned} \mathbf{Lu} - \mathbf{f} &= \mathbf{0} \\ \mathbf{Q} &= \mathbf{Lu} - \mathbf{f}. \end{aligned} \qquad (8)$$

where **Q** is a potential if the equality

$$\left\langle \mathrm{d}\mathbf{Q}(\mathbf{u},\bar{\mathbf{u}}),\mathbf{u}^* \right\rangle = \left\langle \mathrm{d}\mathbf{Q}\left(\mathbf{u},\mathbf{u}^*\right),\bar{\mathbf{u}} \right\rangle. \qquad (9)$$

is satisfied. Where $\mathbf{dQ(u,\bar{u})}$ is the Gâteaux derivative of **Q** and the inner product to two vectors. Therefore, the functional corresponding to the field equations is obtained as

$$\mathrm{I}(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{1}} \left\langle \mathbf{Q}(\mathrm{s}\,\mathbf{u},\mathbf{u}),\mathbf{u} \right\rangle \mathrm{ds}. \qquad (10)$$

where s is a scalar quantity. Details of variational procedures can be found in [17]. Finally from (10), the functional of cross-ply laminated composite thick plates becomes:

$$\begin{aligned} I(y) = {} & \left[\mathrm{M_x},\Omega_{\mathrm{x,x}}\right] + \left[\mathrm{M_{xy}},\Omega_{\mathrm{x,y}}\right] + \left[\mathrm{M_{xy}},\Omega_{\mathrm{y,x}}\right] + \left[\mathrm{M_y},\Omega_{\mathrm{y,y}}\right] \\ & + \left[\mathrm{Q_x},\Omega_{\mathrm{x}}\right] + \left[\mathrm{Q_y},\Omega_{\mathrm{y}}\right] + \left[\mathrm{Q_x},\mathrm{w}_{,\mathrm{x}}\right] + \left[\mathrm{Q_y},\mathrm{w}_{,\mathrm{y}}\right] - [\mathrm{q},\mathrm{w}] \\ & - \frac{\mathrm{Q_{11}}}{2\mathrm{E_x D_{11}}}\left\{\left[\mathrm{M_x},\mathrm{M_x}\right] - 2\mu_{\mathrm{xy}}\left[\mathrm{M_x},\mathrm{M_y}\right] + \left[\mathrm{M_y},\mathrm{M_y}\right] + 2(1+\mu_{\mathrm{xy}})\left[\mathrm{M_{xy}},\mathrm{M_{xy}}\right]\right\} \\ & + \frac{\mathrm{Q_{11}h^2}}{\mathrm{E_x D_{11}10}}\left\{\mu_{\mathrm{xz}}\left[\mathrm{q},\mathrm{M_x}\right] + \mu_{\mathrm{yz}}\left[\mathrm{q},\mathrm{M_y}\right]\right\} - \frac{3}{5\mathrm{G_{xz}h}}\left[\mathrm{Q_x},\mathrm{Q_x}\right] - \frac{3}{5\mathrm{G_{yz}h}}\left[\mathrm{Q_y},\mathrm{Q_y}\right] \\ & - [\mathrm{q},\mathrm{w}] - \left[\hat{\mathrm{M}},\Omega\right]_\sigma - \left[\hat{\mathrm{Q}},\mathrm{w}\right]_\sigma - \left[(\mathrm{w}-\hat{\mathrm{w}}),\mathrm{Q}\right]_\varepsilon - \left[(\Omega-\hat{\Omega}),\mathrm{M}\right]_\varepsilon \end{aligned} \qquad (11)$$

where subscript, σ represents dynamic boundary condition, ε represents geometric boundary condition.

The [q, w] expression in the functional **I(y)** can be defined as below in the dynamic analysis:

$$[q,w]=\frac{1}{2}\rho\omega[w,w]. \qquad (12)$$

Element matrix derived for free vibration analysis, convert to the eigenvalue problem as below:

$$\left([K^{*}]-\omega^{2}[M]\right)\{w\}=\{0\}. \qquad (13)$$

$$[K^{*}]=[K_{22}]-[K_{12}]^{T}[K_{11}]^{-1}[K_{12}]. \qquad (14)$$

where K* represents reduced system matrices. Solution of this equation set is satisfied when coefficient determinant equals to zero.

III. Numerical Example

Example 1:

An orthotropic simply supported cross-ply ($0^0/90^0/0^0/90^0/0^0$) laminated thick plate of which its size to thickness ratio (thickness parameter), 2a/h=10 is considered. To select suitably refined mesh scheme, this example is solved for different orders of mesh scheme.

Based on the different orders of mesh scheme, number of elements increased as shown in the first column of Table I.

Material properties are;

$E_1 = 25E_2$; $G_{12} = G_{13} = 0.5E_2$; $G_{23} = 0.2E_2$; $\mu_{12} = 0.25$

Frequency parameter $\bar{\omega}=\omega\,(2a^2/h)\sqrt{\rho/E_2}$ of cross-ply laminated thick plate for different number of elements are compared with previously published results [16] and presented in Table I. The numerical results show good convergence.

TABLE I. FREQUENCY PARAMETERS $\bar{\omega}$ FOR A CROSS-PLY ($0^0/90^0/0^0/90^0/0^0$) LAMINATED SIMPLY SUPPORTED THICK PLATE FOR DIFFERENT MESH SCHEME

Number of Elements	*Ref [16]*	*Present*
3x3	9,215	9,1732
5x5	9,215	9,1780
7x7	9,215	9,1783
9x9	9,215	9,1784

Example 2:

An orthotropic simply supported cross-ply ($0^0/90^0/0^0$) laminated thick plate of which its size to thickness ratio (thickness parameter), 2a/h=10 is examined. A uniform mesh (10x10) is used in a quarter of the plate.

Material properties are;

$E_1 = 25E_2$; $G_{12} = G_{13} = 0.5E_2$; $G_{23} = 0.2E_2$; $\mu_{12} = 0.25$

In order to determine the dependency of vibration frequencies on boundary conditions, first vibration frequency of cross-ply laminated thick plates with different boundary conditions S-S-S-S (all edges are simply supported), S-S-S-C (three ends are simply supported and one end is clamped) and S-S-C-C (two opposite ends simply supported and other two ends clamped) are computed. Dependency of vibration frequency on the boundary condition is presented in Table II and frequency parameter $\bar{\omega}=\omega\,(2a^2/h)\sqrt{\rho/E_2}$ results are compared with previously published results [9]. The numerical results show good convergence.

TABLE II. DEPENDENCY OF FIRST VIBRATION FREQUENCY PARAMETER FOR DIFFERENT BOUNDARY CONDITIONS

Boundary Condition	*Ref [9]*	*Present*
S-S-S-S	12.163	12.256
S-S-S-C	14.248	14.307
S-S-C-C	16.383	16.405

Example 3:

An orthotropic simply supported cross-ply ($0^0/90^0/90^0/0^0$) laminated thick plate is considered. A uniform mesh (9x9) is used in a quarter of the plate.

Material and geometrical properties are;

$G_{12} = G_{13} = 0.6E_2$; $G_{23} = 0.5E_2$; $\mu_{12} = 0.25$; 2a/h=5

Frequency parameter $\bar{\omega}=\omega\,(2a^2/h)\sqrt{\rho/E_2}$ of cross-ply laminated simply supported thick plate for different E_1/E_2 ratio. The results are compared with previously published results [18] and presented in Table III. As can be seen from the table, the results obtained using the present mixed finite element method agrees closely with available solutions.

TABLE III. FREQUENCY PARAMETER $\bar{\omega}$ OF SIMPLY SUPPORTED CROSS-PLY LAMINATED THICK PLATE FOR DIFFERENT E_1/E_2 RATIO

E_1/E_2	*Ref [18]*	*Present*
10	8.340	8.519
20	9.613	9.798
30	10.372	10.547
40	10.899	11.061

Example 4:

As a last example, vibration frequency of simply supported cross-ply ($0^0/90^0/0^0$) laminated thick plate are determined. A uniform mesh (9x9) is used in a quarter of the plate.

Material and geometrical properties are;

E_1 = 175 GPa; E_2= 7 GPa; $G_{12} = G_{13}$ = 3.5 GPa; G_{23} = 1.4 GPa; μ_{12} = 0.25; 2a/h=10

TABLE IV. VIBRATION FREQUENCY OF SIMPLY SUPPORTED CROSS-PLY ($0^0/90^0/0^0$) LAMINATED THICK PLATE (2A/H=10)

2a/h	*Ref [12]*	*Present*
10	12.190	12.272

The results are compared with previously published results [12] and presented in Table IV. As can be seen from the table, the results obtained using the present mixed finite element method agrees closely with available solutions.

IV. Conclusion

In this study, with Gâteaux differential method a new functional corresponding to free vibration analysis of the cross-ply laminated composite thick plate is presented. For the analysis, mixed finite element method is used to obtain element matrices. Functional of cross-ply laminated thick plate has eight unknowns in each node. For the solutions of functional, linear shape functions are used because of only first-order derivation exists in functional.

Employing the developed mixed finite element formulation, cross-ply laminated composite thick plates including different number of layers are taken as examples to numerically evaluate the effects of the variation of geometrical and material parameters, and boundary conditions on the vibration frequencies.

The performance of the presented mixed finite element formulation is verified by comparing the obtained results with the results of the numerical examples in the literature. The numerical results are found to be in good agreement with those available in the literature.

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