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Free Vibration Analysis of Elastic Cylindrical Helix via a Mixed Finite Element Method

Nihal Eratlı, Akif Kutlu

Abstract—**Referring to the previous theoretical and numerical studies on the static/dynamic analyses of elastic helicoidal bars, only circular or square cross-sections were considered up today. The main purpose of this study is to investigate the dynamic behavior of the elastic cylindrical helices having hollow circle and elliptical hollow sections. For this purpose, a mixed finite element formulation with the Timoshenko beam assumptions is employed. Frenet triad is adopted as the local coordinate system in the helix geometry. Isoparametric curved elements involve two nodes, where each node has 12 DOF, namely three translations, three rotations, two shear forces, one axial force, two bending moment and one torque. Numerical solutions are performed to analyze the free vibration behavior of the cylindrical helices and benchmark results are presented. Parametric studies are carried out to investigate the influence of the parameters e.g. section geometry, boundary conditions, density of the material and number of turns.**

Keywords—**cylindrical helix, elliptical hollow section, mixed finite element method, free vibration**

I. Introduction

Helix type structural elements are frequently used in many engineering applications, such as, helicoidal staircases in civil engineering and vehicle suspension systems in mechanical engineering. In the literature, although a tremendous amount of theoretical and numerical studies exist on the static/dynamic analyses of elastic helixes, it can observed that only beams with circular or square cross-sections were considered ([1]-[8]). Analytical study of Yu and Hao [9] considered the warping deformations of the cross-section in the free vibration analysis of cylindrical helical springs with noncircular cross-sections. In [10], the free vibration analysis of cylindrical helix with elliptical cross-section is investigated and verified numerically using the problem existing in [9].

It is straightforward to calculate the torsional rigidity of bars having a circular cross section. In the case of other geometries some special treatments have to be shown to determine the torsional rigidity. It is possible to find some exact formulas ([11]) in the open literature expressing the torsional moment of inertia for various arbitrary cross-sections e.g. ellipse and equilateral triangle, some approximated analytical formulas ([12]) and numerical solution procedures ([13]-[15]).

Nihal Eratlı, Akif Kutlu

Department of Civil Engineering, Istanbul Technical University
Turkey

This study performs free vibration analysis of elastic cylindrical helices having hollow circle and elliptical hollow sections. For this purpose a mixed finite element formulation comprising the Timoshenko beam theory is employed. The formulation used for the torsional rigidity of the elliptical hollow section is given in [12]. Parametric studies are reported to analyze the free vibration behavior of cylindrical helices with circular and elliptical hollow sections. First five frequencies cylindrical helices are presented for various numbers of turns, material densities, geometric properties and boundary conditions.

II. Formulation

A. *The Field Equations and the Functional*

The details of the field equations for the elastic cylindrical helix based on Timoshenko beam theory can be found in [2] and [16]. For the completeness of the paper they will be introduced briefly here. Thus, the referred equations can be listed as follows:

Equations of motion:

$$\left.\begin{aligned} -\mathbf{T}_{,s}-\mathbf{q}+\rho A\ddot{\mathbf{u}}=\mathbf{0} \\ -\mathbf{M}_{,s}-\mathbf{t}\times\mathbf{T}-\mathbf{m}+\rho\mathbf{I}\ddot{\mathbf{\Omega}}=\mathbf{0} \end{aligned}\right\} \quad (1)$$

Kinematic equations:

$$\left.\begin{aligned} \mathbf{u}_{,s}+\mathbf{t}\times\mathbf{\Omega}-\boldsymbol{\gamma}=\mathbf{0} \\ \mathbf{\Omega}_{,s}-\boldsymbol{\kappa}=\mathbf{0} \end{aligned}\right\} \quad (2)$$

Constitutive equations:

$$\left.\begin{aligned} \boldsymbol{\gamma}-\mathbf{C}_{\gamma}\mathbf{T}=\mathbf{0} \\ \boldsymbol{\kappa}-\mathbf{C}_{\kappa}\mathbf{M}=\mathbf{0} \end{aligned}\right\} \quad (3)$$

Here $\mathbf{u}=u_t\,\mathbf{t}+u_n\,\mathbf{n}+u_b\,\mathbf{b}$ is the displacement vector, $\mathbf{\Omega}=\Omega_t\,\mathbf{t}+\Omega_n\,\mathbf{n}+\Omega_b\,\mathbf{b}$ is the cross section rotation vector.

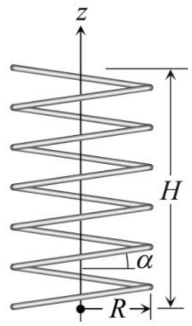

Figure 1. Cylindrical helix geometry.

$\mathbf{T}=T_t\,\mathbf{t}+T_n\,\mathbf{n}+T_b\,\mathbf{b}$ defines the force vector, $\mathbf{M}=M_t\,\mathbf{t}+M_n\,\mathbf{n}+M_b\,\mathbf{b}$ is the moment vector, ρ is the material density. A is the area of the cross section, $\mathbf{I}$ stores the moment of inertia, $\boldsymbol{\gamma}$ is the unit shear vector, $\boldsymbol{\kappa}$ is the unit rotation vector, $\mathbf{C}_\gamma$ and $\mathbf{C}_\kappa$ are the compliance matrices. $\mathbf{q}$ and $\mathbf{m}$ are the distributed external force and moment vectors, respectively. Incorporating Gâteaux differential with potential operator concept ([17]) yields the functional in terms of (1)-(3) as,

$$\left.\begin{aligned}
&\mathbf{I}(\mathbf{y})=-\left[\mathbf{u},\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right]+\left[\mathbf{t}\times\boldsymbol{\Omega},\mathbf{T}\right]-\left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}s},\boldsymbol{\Omega}\right]-\tfrac{1}{2}\left[\mathbf{C}_\kappa\ \mathbf{M},\mathbf{M}\right]\\
&-\tfrac{1}{2}\left[\mathbf{C}_\gamma\ \mathbf{T},\mathbf{T}\right]-\tfrac{1}{2}\rho A\omega^2\left[\mathbf{u},\mathbf{u}\right]-\tfrac{1}{2}\rho\,\omega^2\left[\mathbf{I}\,\boldsymbol{\Omega},\boldsymbol{\Omega}\right]\\
&-\left[\mathbf{q},\mathbf{u}\right]-\left[\mathbf{m},\boldsymbol{\Omega}\right]+\left[\left(\mathbf{T}-\hat{\mathbf{T}}\right),\mathbf{u}\right]_\sigma+\left[\left(\mathbf{M}-\hat{\mathbf{M}}\right),\boldsymbol{\Omega}\right]_\sigma\\
&+\left[\hat{\mathbf{u}},\mathbf{T}\right]_\varepsilon+\left[\hat{\boldsymbol{\Omega}},\mathbf{M}\right]_\varepsilon
\end{aligned}\right\}\quad(4)$$

where the square brackets indicate the inner product, the terms with hats in (4) are known values on the boundary and the subscripts ε and σ represent the geometric and dynamic boundary conditions, respectively. Once the motion is considered as harmonic for the free vibration of the helix, conditions $\mathbf{q}=\mathbf{m}=\mathbf{0}$ are satisfied. For further access to the detailed formulation, the reader is referred to the paper [2].

B. *Mixed Finite Element Formulation*

Helicoidal beam is discretized with curved bar elements having two nodes with 12 degrees of freedom at each node. Linear shape functions $\phi_i=(\theta_j-\theta)/\Delta\theta$ and $\phi_j=(\theta-\theta_i)/\Delta\theta$ are used to interpolate field variables over the problems spatial domain. The subscripts i and j represent node numbers of the bar element and $\Delta\theta=(\theta_j-\theta_i)$. The detailed formulation of the mixed finite element matrices considering the variation of helix geometry can be found in [2] and [16]. Field variables at a node can be collected in a vector form as

$$\mathbf{X}^T=\{u_t,u_n,u_b,\Omega_t,\Omega_n,\Omega_b,T_t,T_n,T_b,M_t,M_n,M_b\}\quad(5)$$

C. *Dynamic Analysis in the Mixed Finite Element Formulation*

Calculation of the natural free vibration frequencies of a structural system yields to the following standard eigenvalue problem,

$$\left([\mathbf{K}]-\omega^2[\mathbf{M}]\right)\{\mathbf{u}\}=\{\mathbf{0}\}\quad(6)$$

Here, $[\mathbf{K}]$ and $[\mathbf{M}]$ are the system and mass matrix of the entire domain, respectively. $\mathbf{u}$ is the eigenvector (mode shape) and ω depicts the natural angular frequency of the system. The compact form of mixed finite element matrices in (6) can be expanded as follows,

$$\left(\begin{bmatrix}[\mathbf{K}_{11}] & [\mathbf{K}_{12}]\\ [\mathbf{K}_{22}] & [\mathbf{K}_{22}]\end{bmatrix}-\omega^2\begin{bmatrix}[\mathbf{0}] & [\mathbf{0}]\\ [\mathbf{0}] & [\mathbf{M}]\end{bmatrix}\right)\begin{Bmatrix}\{\mathbf{F}\}\\ \{\mathbf{U}\}\end{Bmatrix}=\begin{Bmatrix}\{\mathbf{0}\}\\ \{\mathbf{0}\}\end{Bmatrix}\quad(7)$$

Here, $\{\mathbf{F}\}$ corresponds to the nodal force and moment vectors, $\{\mathbf{U}\}=\{\mathbf{u}\ \boldsymbol{\Omega}\}^T$ contains the nodal displacement and rotation vectors. In order to reduce the problem from the mixed form in (7) to a standard eigenvalue form as in (6), $\{\mathbf{F}\}$ vector is eliminated from (7) by a condensation procedure, which produces the condensed system matrix $[\mathbf{K}^*]=[\mathbf{K_{22}}]-[\mathbf{K_{12}}]^{\mathrm{T}}[\mathbf{K_{11}}]^{-1}[\mathbf{K_{12}}]$. Finally, the eigenvalue problem in the mixed formulation yields to the form,

$$\left([\mathbf{K}^*]-\omega^2[\mathbf{M}]\right)\{\mathbf{U}\}=\{\mathbf{0}\}\quad(8)$$

III. Numerical Examples

The free vibration analysis of the cylindrical helix problem with fixed-fixed and fixed-free boundary conditions is investigated in order to compare the performance of hollow circle and two different elliptical with hollow cross-sections, where all the cross-sections are chosen to have the same constant area. The effects of the number of active turns, the material density are also considered. The selected material and geometrical properties for the helicoidal beam are: the modulus of elasticity $E=210\,\mathrm{GPa}$, Poisson's ratio $\upsilon=0.3$, the material densities $\rho=7850\,\mathrm{kg/m^3}$ and $\rho=8500\,\mathrm{kg/m^3}$, the number of active turns $n=3.5$, 7.5, 11.5, the vertical height and radius of the cylindrical helix $H=0.6\,\mathrm{m}$ and $R=0.2\,\mathrm{m}$, respectively (see Figure 1). The outer and inner radii of hollow circle cross-section $r_1=15\,\mathrm{mm}$ and $r_2=7.5\,\mathrm{mm}$, respectively. The major radii of two different elliptical hollow cross-sections are $a_1=25\,\mathrm{mm}$ and $a_1=40\,\mathrm{mm}$, by keeping $b_1=15\,\mathrm{mm}$ constant, respectively.

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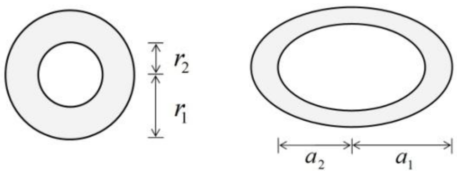

Figure 2. Circular and elliptical hollow sections.

For two different elliptical hollow cross-sections, the inner radius to the outer radius ratios of the minor and major radii to the major radius ratios are taken $b_2 / b_1 = a_2 / a_1 = 0.74$ and 0.85, respectively (see Figure 2). The first five natural frequencies of cylindrical helix for all cross-sections are tabulated in Table I.

The results obtained by using 200 finite elements can be summarized as follows:

- The natural frequencies of cylindrical helix with elliptical with hollow cross-sections are greater than the natural frequencies of cylindrical helix with the circular cross-section.
- In the case of the fixed-fixed boundary condition, with respect to the fundamental natural frequencies in Table 1 for the hollow circle section, the percent increases for the elliptical with hollow-1 and the elliptical with hollow-2 sections are 34%□36% and 54%□56%, respectively. Similarly, in the case of the fixed-free boundary condition, with respect to the fundamental natural frequencies for the hollow circle section, the percent increases for the elliptical with hollow-1 and the elliptical with hollow-2 sections are 27%□29% and 38%□42%, respectively.
- If the fundamental natural frequencies are compared with the results that correspond to the elliptical with hollow-1, the percent increases in the fundamental natural frequency, which corresponds to the elliptical with hollow-2 for the fixed-fixed and the fixed-free boundary conditions, ranges from 14%□15% and 7%□12%, respectively.
- As n increase, the natural frequencies of the cylindrical helix reduce, the closer frequencies are obtained and coupled modes begin to appear. For the all boundary conditions, the discussed sections and the material densities, with respect to the fundamental natural frequencies for the number of turns $n = 3.5$, the percent decreases for $n = 7.5$ and $n = 11.5$ are approximately □52% and □69%, respectively.
- The natural frequencies are inversely proportional with the material density ρ. In the case of the fixed-fixed boundary condition, with respect to the fundamental natural frequencies for the material density $\rho = 7850\,\text{kg/m}^3$, the percent decreases for the material density $\rho = 8500\,\text{kg/m}^3$ are 3%□4%. Similarly, in the case of the fixed-free boundary condition, with respect to the fundamental natural frequencies for the material density $\rho = 7850\,\text{kg/m}^3$, the percent decreases for the material density $\rho = 8500\,\text{kg/m}^3$ are 2%□7%.

Table I. The first five frequencies (in Hz) of the cylindrical helix with hollow circle and two different elliptical hollow cross-sections for different parameters (the number of turns n, the material density ρ, the boundary conditions)

n	hollow circle ($r_2 / r_1 = 0.5$)				elliptical with hollow-1 ($b_2 / b_1 = a_2 / a_1 = 0.74$)				elliptical with hollow-2 ($b_2 / b_1 = a_2 / a_1 = 0.85$)			
	fixed-fixed		fixed-free		fixed-fixed		fixed-free		fixed-fixed		fixed-free	
	$\rho(\text{kg/m}^3)$		$\rho(\text{kg/m}^3)$		$\rho(\text{kg/m}^3)$		$\rho(\text{kg/m}^3)$		$\rho(\text{kg/m}^3)$		$\rho(\text{kg/m}^3)$	
	7850	8500	7850	8500	7850	8500	7850	8500	7850	8500	7850	8500
3.5	21.0	20.2	7.0	6.8	28.3	27.2	8.9	8.6	32.4	31.2	9.9	9.6
	22.7	21.8	7.0	6.8	38.5	37.0	9.1	8.7	47.1	45.2	10.1	9.8
	27.1	26.0	10.6	10.2	39.6	38.1	14.3	13.7	47.4	45.6	16.4	15.7
	28.1	27.0	12.1	11.6	45.0	43.3	22.4	21.5	60.3	58.0	36.5	35.1
	37.7	36.3	23.5	22.6	52.6	50.5	33.1	31.8	68.6	65.9	38.8	37.3
7.5	10.0	9.6	3.3	3.2	13.5	13.0	4.2	4.1	15.5	14.9	4.7	4.5
	11.3	10.8	3.3	3.2	19.4	18.6	4.2	4.1	23.8	22.8	4.7	4.6
	13.1	12.6	5.0	4.8	19.6	18.9	6.8	6.5	23.8	22.8	7.8	7.5
	13.3	12.8	5.7	5.5	21.4	20.5	10.6	10.2	30.5	29.3	17.7	17.0
	19.7	18.9	11.4	11.0	26.6	25.5	16.1	15.4	35.3	33.9	18.9	18.1
11.5	6.5	6.3	2.2	2.1	8.8	8.5	2.8	2.7	10.1	9.7	3.1	2.9
	7.4	7.1	2.2	2.1	12.8	12.3	2.8	2.7	15.7	15.0	3.1	3.0
	8.6	8.3	3.3	3.1	12.9	12.4	4.4	4.2	15.7	15.0	5.1	4.9
	8.7	8.3	3.7	3.6	13.9	13.4	6.9	6.7	20.1	19.4	11.6	11.1
	13.0	12.5	7.5	7.2	17.5	16.9	10.6	10.1	23.2	22.3	12.4	11.9

- As expected, the natural frequencies of cylindrical helix with fixed-fixed boundary condition are greater than fixed-free boundary condition. For the discussed sections, the number of turns and the material densities, with respect to the fundamental natural frequencies for the fixed-fixed boundary condition, the percent decreases for the fixed-free boundary condition are 66% – 70%.

IV. Conclusions

This study investigates the free vibration response of cylindrical helicoidal beams. For this purpose a two-field mixed finite element formulation depending on the potential operator concept and Gâteaux differential is employed, where Timoshenko beam assumptions are adopted to the beam theory. Two nodded curved finite elements with bilinear shape functions are used to discretize the spatial domain of the problem. Eigen-value equations of the dynamic problem are obtained after a condensation procedure by elimination stress type terms from the finite element matrices. Some parametric studies are performed to observe the effects of material density, number of turns, boundary conditions and ellipticity of hollow cross section on the free vibration characteristics of the cylindrical helicoidal beam structures.

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References

[1] J.E. Mottershead, Finite elements for dynamical analysis of helical rods, Int. J. Mech. Sci. 22 (1980) 267–283. doi:10.1016/0020-7403(80)90028-4.
[2] M.H. Omurtag, A.Y. Aköz, The mixed finite element solution of helical beams with variable cross-section under arbitrary loading, Comput. Struct. 43 (1992) 325–331. doi:10.1016/0045-7949(92)90149-T.
[3] V. Haktanir, E. Kiral, Statical analysis of elastically and continuously supported helicoidal structures by the transfer and stiffness matrix methods, Comput. Struct. 49 (1993) 663–677. doi:10.1016/0045-7949(93)90070-T.
[4] V. Yildirim, Investigation of parameters affecting free vibration frequency of helical springs, Int. J. Numer. Methods Eng. 39 (1996) 99–114. doi:10.1002/(SICI)1097-0207(19960115)39:1<99::AID-NME850>3.0.CO;2-M.
[5] S.A. Alghamdi, M.A. Mohiuddin, H.N. Al-Ghamedy, Analysis of free vibrations of helicoidal beams, Eng. Comput. 15 (1998) 89–102. doi:10.1108/02644409810200712.
[6] W. Busool, M. Eisenberger, Exact static analysis of helicoidal structures of arbitrary shape and variable cross section, J. Struct. Eng. 127 (2001) 1266–1275. doi:10.1061/(ASCE)0733-9445(2001)127:11(1266).
[7] J. Lee, Free vibration analysis of cylindrical helical springs by the pseudospectral method, J. Sound Vib. 302 (2007) 185–196. doi:10.1016/j.jsv.2006.11.008.
[8] A.M. Yu, C.J. Yang, G.H. Nie, Analytical formulation and evaluation for free vibration of naturally curved and twisted beams, J. Sound Vib. 329 (2010) 1376–1389. doi:10.1016/j.jsv.2009.11.014.
[9] A.M. Yu, Y. Hao, Free vibration analysis of cylindrical helical springs with noncircular cross-sections, J. Sound Vib. 330 (2011) 2628–2639. doi:10.1016/j.jsv.2010.12.015.
[10] M. Ermiş, N. Eratlı, M.H. Omurtag, Static and dynamic analysis of elastic cylindrical helixes with arbitrary cross-sections by using the mixed finite element method, in: 11th Int. Congr. Adv. Civ. Eng. 21-25 Oct., İstanbul, 2014, (accepted).
[11] S. Timoshenko, J.N. Goodier, Theory of elasticity, McGraw-Hill, New York, 1951.
[12] N.X. Arutunan, B.L. Abraman, Torsion of Elastic Bodies (in Russian), State Publisher, Fizmatgiz, Moscow, 1963.
[13] J.F. Ely, O.C. Zienkiewicz, Torsion of compound bars—A relaxation solution, Int. J. Mech. Sci. 1 (1960) 356–365. doi:10.1016/0020-7403(60)90055-2.
[14] E.J. Sapountzakis, Nonuniform torsion of multi-material composite bars by the boundary element method, Comput. Struct. 79 (2001) 2805–2816. doi:10.1016/S0045-7949(01)00147-X.
[15] L.R. Herrmann, Elastic torsional analysis of irregular shapes, J. Eng. Mech. Div. 91 (1965) 11–20.
[16] K. Girgin, Free vibration analysis of non-cylindrical helices with, variable cross-section by using mixed FEM, J. Sound Vib. 297 (2006) 931–945. doi:10.1016/j.jsv.2006.05.001.
[17] J.T. Oden, J.N. Reddy, Variational methods in theoretical mechanics, Springer-Verlag, 1976.

About Authors:

Nihal Eratlı
Assoc.Prof.Dr.

Akif Kutlu
Dr.