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Study on improving the theory of plate bending

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Abstract—**This paper presents study on improving the theory of plate bending. In doing this, it was assumed that none of the components of stress tensor and strain tensor shall be neglected. This assumption presupposes that all the components of stress tensor and strain tensor are significant. Constitutive equations were determined. Equilibrium equations of a static body at state of stress was used in conjunction with equations of strain-displacement and constitutive equations to determine the improved governing equation and stress resultants. A numerical problem was solved for a square plate simply supported at all the four edges. Obtained result indicated that maximum deflection and bending moment are -0.003377qa^4/D and -0.0397125qa^2 respectively. The corresponding values from classical theory are 0.00414qa^4/D and 0.05163qa^2 respectively. It was observed that the maximum deflection and bending moment from classical theory were higher than the magnitude of those from the present study by 18% and 23% respectively. It was also observed that the values from the present study are negative while those from the classical theory are positive.**

***Keywords*—Plate bending, stress tensor, strain tensor, constitutive equation, equilibrium equation, governing equation, stress resultants, deflection, bending moment**

I. Introduction

Some of Kirchhoff's or classical plate theory assumptions [1, 2, 3] are:

i. Vertical shear strains γ_{xz} and γ_{yz} and normal vertical strain ε_z are negligible and should be omitted from the strain tensor.
ii. Normal vertical stress σ_z is negligible and should be omitted from the stress tensor.

These assumptions, no doubt simplified the bending theory of plates but seriously over estimates the bending stresses. This drawback attracted the attention of earlier Scholars [4, 5, 6, 7] who tried to refine the plate bending theory. However, their works presented complex and complicated governing equations and equations of stress resultants. These refined theories of plate bending have received little patronage as most recent works on plates had depended on the classical plate theories. Under this scenario, one wonders whether, the plate theory can be improved without arriving at complexities. Thus, the main objective of this study is to develop an improved theory of plate bending by dropping the controversial assumptions. In doing this the following assumptions shall be adopted:

i. The plate is isotropic, homogeneous and initially flat.
ii. A straight line normal to the middle surface of the plate remains straight and normal to middle surface after bending.
iii. For thin plate, normal vertical strain εz is constant. That is derivatives of normal vertical strain εz are zero. For thick plate, normal vertical strain εz varies along the depth of the plate.

Strain tensors
The displacements along x, y and z axes are denoted as u, v and w respectively. Of common knowledge are following components strain tensors.

II. Strain tensors

The displacements along x, y and z axes are denoted as u, v and w respectively. Of common knowledge are following components strain tensors.

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (1)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (2)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad (3)$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\gamma_{xy}}{2} \quad (4)$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} = \frac{\gamma_{xz}}{2} \quad (5)$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} = \frac{\gamma_{yz}}{2} \quad (6)$$

III. Basic constitutive equations

The six basic constitutive equations are stated:

$$\sigma_x - \mu\sigma_y - \mu\sigma_z = E\varepsilon_x \quad (7)$$

$$\sigma_y - \mu\sigma_x - \mu\sigma_z = E\varepsilon_y \quad (8)$$

$$\sigma_z - \mu\sigma_x - \mu\sigma_y = E\varepsilon_z \quad (9)$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} \quad (10)$$

$$\tau_{xz} = \frac{E}{2(1+\mu)}\gamma_{xz} \quad (11)$$

$$\tau_{yz} = \frac{E}{2(1+\mu)}\gamma_{yz} \quad (12)$$

Rearranging equation (9) gives:

$$\sigma_z = E\varepsilon_z + \mu\sigma_x + \mu\sigma_y \quad (13)$$

Substituting equation (13) into equations (7) and (8) gives:

$$\sigma_x(1-\mu) - \mu\sigma_y = \frac{E}{1+\mu}(\varepsilon_x + \mu\varepsilon_z) \quad (14)$$

$$\sigma_y(1-\mu) - \mu\sigma_x = \frac{E}{1+\mu}(\varepsilon_y + \mu\varepsilon_z) \quad (15)$$

Solving equations (14) and (15) simultaneously gives:

$$\sigma_x = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\varepsilon_x + \mu\varepsilon_y + \mu\varepsilon_z\right] \quad (16)$$

$$\sigma_y = \frac{E}{(1+\mu)(1-2\mu)}\left[(1-\mu)\varepsilon_y + \mu\varepsilon_x + \mu\varepsilon_z\right] \quad (17)$$

IV. Equations of equilibrium

The three body forces of finite element acting along x, y and z axes are denoted as FX, Fy and Fz.

For an element in a state of stress, the three basic equations of static equilibrium are of common knowledge and they are:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}=F_x \tag{18}$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yz}}{\partial z}=F_y \tag{19}$$

$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}=F_z \tag{20}$$

V. Improved theory equations

For a plate subject to uniformly distributed load normal to the middle surface of the plate, Fx = Fy = 0. With this assumption, equations (18) and (19), upon rearrangement and integration become:

$$\tau_{xz}=-\frac{\partial \sigma_x}{\partial x}z-\frac{\partial \tau_{xy}}{\partial y}z \tag{21}$$

$$\tau_{yz}=-\frac{\partial \sigma_y}{\partial y}z-\frac{\partial \tau_{xy}}{\partial x}z \tag{22}$$

Substituting, equations (13), (21) and (22) into equation (20) gives:

$$F_z=E\frac{\partial \varepsilon_z}{\partial z}+\mu\frac{\partial \sigma_x}{\partial z}+\mu\frac{\partial \sigma_y}{\partial z}-\frac{\partial^2 \sigma_x}{\partial x^2}z-2\frac{\partial^2 \tau_{xy}}{\partial x\partial y}z-\frac{\partial^2 \sigma_y}{\partial y^2}z \tag{23}$$

From equation (5) we obtain:

$$\partial u=\frac{\partial w}{\partial x}\partial z \tag{24}$$

Integrating equation (24) gives:

$$u=z\frac{\partial w}{\partial x}+u_0 \tag{25}$$

Similarly, from equation (6) we obtain:

$$v=z\frac{\partial w}{\partial y}+v_0 \tag{26}$$

Differentiating equation (25) gives:

$$\varepsilon_x=z\frac{\partial^2 w}{\partial x^2} \tag{27}$$

$$\gamma_{xy}=2z\frac{\partial^2 w}{\partial x\partial y} \tag{28}$$

Similarly, differentiating equation (26) gives:

$$\varepsilon_y=z\frac{\partial^2 w}{\partial y^2} \tag{29}$$

$$\gamma_{xy}=2z\frac{\partial^2 w}{\partial x\partial y} \tag{30}$$

Substituting equations (3), (27) and (29) into equations (16) and (17) gives:

$$\sigma_x=\frac{E}{(1+\mu)(1-2\mu)}[z(1-\mu)\frac{\partial^2 w}{\partial x^2}+z\mu\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial w}{\partial z}] \tag{31}$$

$$\sigma_y=\frac{E}{(1+\mu)(1-2\mu)}\left[z(1-\mu)\frac{\partial^2 w}{\partial y^2}+z\mu\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial w}{\partial z}\right] \tag{32}$$

Substituting equation (30) into equations (10) gives:

$$\tau_{xy}=\frac{Ez}{(1+\mu)}\frac{\partial^2 w}{\partial x\partial y} \tag{33}$$

Substituting equations (31), (32) and (33) into equations (23) gives:

$$F_z=E\frac{\partial^2 w}{\partial z^2}+T\mu\{[z(1-\mu)\frac{\partial^3 w}{\partial x^2\partial z}+z\mu\frac{\partial^3 w}{\partial y^2\partial z}+\mu\frac{\partial^2 w}{\partial z^2}]+[z(1-\mu)\frac{\partial^3 w}{\partial y^2\partial z}+z\mu\frac{\partial^3 w}{\partial x^2\partial z}+\mu\frac{\partial^2 w}{\partial z^2}]\}+T\{[-z^2(1-\mu)\frac{\partial^4 w}{\partial x^4}-z^2\mu\frac{\partial^4 w}{\partial x^2\partial y^2}-\mu z\frac{\partial^3 w}{\partial x^2\partial z}]-2z^2(1-2\mu)\frac{\partial^4 w}{\partial x^2\partial y^2}[-z^2(1-\mu)\frac{\partial^4 w}{\partial y^4}-\mu z^2\frac{\partial^4 w}{\partial x^2\partial y^2}-\mu z\frac{\partial^3 w}{\partial y^2\partial z}]\}$$

Where $T=\frac{E}{(1+\mu)(1-2\mu)}$

Adding like terms together gives:

$$F_z=E\frac{\partial^2 w}{\partial z^2}+T\mu[z\frac{\partial^3 w}{\partial x^2\partial z}+2\mu\frac{\partial^2 w}{\partial z^2}+z\frac{\partial^3 w}{\partial y^2\partial z}]T[-z^2(1-\mu)\frac{\partial^4 w}{\partial x^4}-2z^2(1-\mu)\frac{\partial^4 w}{\partial x^2\partial y^2}-z^2(1-\mu)\frac{\partial^4 w}{\partial y^4}-\mu z\frac{\partial^3 w}{\partial x^2\partial z}-\mu z\frac{\partial^3 w}{\partial y^2\partial z}] \tag{34}$$

But the uniform distributed load, q normal to the middle surface of the plate is given as:

$$q=\int_{-\frac{t}{2}}^{\frac{t}{2}}F_z\,dz \tag{35}$$

Substituting equation (34) into equation (35) gives:

$$q=Et\frac{\partial^2 w}{\partial z^2}+T\mu\left[0+2\mu t\frac{\partial^2 w}{\partial z^2}+0\right]T[-\frac{t^3(1-\mu)}{12}\frac{\partial^4 w}{\partial x^4}-\frac{t^3(1-\mu)}{6}\frac{\partial^4 w}{\partial x^2\partial y^2}-\frac{t^3(1-\mu)}{12}\frac{\partial^4 w}{\partial y^4}-0-0]$$

That is

$$q=Et\frac{\partial^2 w}{\partial z^2}+\frac{2E\mu^2 t}{(1+\mu)(1-2\mu)}\frac{\partial^2 w}{\partial z^2}+\frac{-Et^3(1-\mu)}{(1+\mu)(1-2\mu)}\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right] \tag{36}$$

Simplifying equation (36) give:

$$q = \frac{12Et^3(1-\mu)^2}{12(1-\mu^2)(1-2\mu)t^2}\left[\frac{\partial^2 w}{\partial z^2}\right] + \frac{-Et^3(1-\mu)^2}{(1-\mu^2)(1-2\mu)}\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \quad (37)$$

The plate flexural rigidity denoted as D is given as:

$$D = \frac{Et^3}{12(1-\mu^2)} \quad (38)$$

Substituting equation (38) into equation (37) gives:

$$q = -BD\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - \frac{12}{t^2}\frac{\partial^2 w}{\partial z^2}\right] \quad (39)$$

Where $B = \frac{(1-\mu)^2}{(1-2\mu)}$

Equation (39) is the governing equation of a thick plate. For thin plate, the last term in equation (39) becomes zero and the governing equation becomes:

$$q = -\frac{(1-\mu)^2}{(1-2\mu)}D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] \quad (40)$$

Stress resultants are given as:

$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_x \, dz \quad (41)$$

$$M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y \, dz \quad (42)$$

$$M_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xy} \, dz \quad (43)$$

$$Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz} \, dz \quad (44)$$

$$Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz} \, dz \quad (45)$$

Substituting equation (31) into equation (41) gives:

$$M_x = \frac{(1-\mu)}{(1-2\mu)}D\left[(1-\mu)\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right] \quad (46)$$

Substituting equation (32) into equation (42) gives:

$$M_y = \frac{(1-\mu)}{(1-2\mu)}D\left[(1-\mu)\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right] \quad (47)$$

Substituting equation (33) into equation (43) gives:

$$M_{xy} = (1-\mu)D\frac{\partial^2 w}{\partial x \partial y} \quad (48)$$

Substituting equation (21), (31) and (33) into equation (44) gives:

$$Q_x = \frac{(1-\mu)^2}{(1-2\mu)}D\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right] \quad (49)$$

Substituting equation (22), (32) and (33) into equation (45) gives:

$$Q_y = \frac{(1-\mu)^2}{(1-2\mu)}D\left[\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\right] \quad (50)$$

VI. Numerical problem

Let's determine the maximum deflection and bending moment of a square plate simply supported at all the four edges (ssss) using this improved theory and assume the Poisson's ratio is 0.30. The deflection equation of ssss plate was given [8] as: $w = A(R - 2R^3 + R^4)(Q - 2Q^3 + Q^4)$

VII. Results

The values of maximum deflection and bending moment from the present study and classical theory are presented on table I.

TABLE I. VALUES OF MAXIMUM DEFLECTION AND BENDING MOMENT

	Maximum deflection	Maximum moment
Improved theory	$-0.003377q\frac{a^4}{D}$	$-0.0397125qa^2$
Classical theory	$0.00414q\frac{a^4}{D}$	$0.05163qa^2$
Percentage difference	18.43%	23%

Looking at table I, one will see that the classical theory of plate bending overestimated the deflection and bending moment by 18.43% and 23% respectively. These values of percentage difference are quite large to be overlooked. With this, one can make good saving of engineering materials during design, construction and fabrication. An observation here is that where as the data from the present study are negative, those from classical theory are positive. This stems from the assumption that the vertical shear strains γ_{xz} and γ_{yz} are not zero. The classical theory assumed them to be zero. Thus, it is recommended that this outcome of this present study be adopted for analysis of plates. It is simply like the classical theory. Unlike the refined theories, it did not present any complexity.

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