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Rough Projective Module

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Abstract—**In recent years, the algebraic structures of rough set theory has been rapidly developed. This paper concerns a relationship between rough sets and projective module. We shall introduce the notion of rough projective module, which is an extended notion of projective module.**

Keywords—rough set theory, rough group, rough ring, rough module.

I. Introduction

Information on the surrounding world is imprecise, incomplete or uncertain. Still our way of thinking and concluding depends on information at our disposal. This means that to draw conclusions, we should able to process uncertain and/or incomplete information. To analyze any type of information, mathematical logic are most appropriate, so we should have to generalize the algebraic structures and logic in the sense of vague or imprecise. Many algebraic structures have been developed over precise set to deal the exact situations. But very few algebraic structures and logics are available to deal with imprecise or vague situations mathematically. Rough set theory is a powerful mathematical tool to handle imprecise situations and rough algebraic structures can play a vital role to handle such situations.

In Pawlak rough set theory, the key concept is an equivalence relation and the building blocks for the construction of the lower and upper approximations are the equivalence classes. The lower approximation of the given set is the union of all the equivalence classes which are the subsets of the set, and the upper approximation is the union of all the equivalence classes which have a non-empty intersection with the set. The object of the given universe can be divided into three classes with respect to any subset

(1) the objects, which are definitely in A;

(2) the objects, which are definitely not in A;

(3) the objects, which are possibly in A;

The objects in class (1) form the lower approximation o A, and the objects in class (1) and (3) together form its upper approximation. The boundary or A is defined as the set of objects in class (2). Z. Bonikowaski introduced the algebraic structures of rough sets [24].

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R. Biswas and S. Nanda introduced the concept of rough group and rough subgroup [13]. N. Kuroki studied the rough ideals in semigroups [10]. B. Davvaz introduced the roughness in rings [1]. B, Davvaz, M. Mahdavipour introduced the roughness in Modules [2]. Rough modules and their some properties are also studied by Zhang Qun-Feng et al. [25]. Standard sources for the algebraic theory of modules are [11, 6]. One can find more on rough set and their algebraic structures in [2, 3, 7, 9, 12, 17, 21, 22]. In recent years, there has been a fast growing intrest in this new emerging theory, ranging frame work in pure theory, such as algebraic foundations and mathematical logic [26-29] to diverse areas of applications.

The aim of this paper is to investigate the Rough Projective Module. The rest of the paper is organized as follows: In section 2, preliminaries are given. In section 3. We introduce the concept of rough projective module. Finally, our conclusions are presented. We have used standard mathematical notation through-out the paper and we assume that the reader is familiar with basic notions of algebra and rough set theory.

II. Preliminaries

In this section, we give some basic definitions of rough algebraic structures and results which will be used later on.

Definition 2.1: [20] A pair (U, θ), where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space.

Definition 2.2: [1]For an approximation space (U, θ), by a rough approximation operator in (U, θ) we mean a mapping $Apr: P(U) \to P(U) \times P(U)$ defined by

$$Apr(X) = (\underline{X}, \overline{X}) \text{ for every } X \in P(U)$$

Where $\underline{X} = \{x \in X | [x]_\theta \subseteq X\}$, $\overline{X} = \{x \in X | [x]_\theta \cap X \neq \emptyset\}$.
$\underline{X}$ is called the lower rough approximation of X in (U, θ), and $\overline{X}$ is called upper rough approximation of X in(U, θ).

Definition 2.3: [1] given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough set in (U, θ) iff $(A, B) = Apr(X)$ for some $X \in P(U)$.

Example 2.1: Let (U, θ) is an approximation space, where $U = \{o_1, o_2, o_3, \cdots, o_7\}$ and an equivalence relation θ with the following equivalence classes:

$$E_1 = \{o_1, o_4\}$$
$$E_2 = \{o_2, o_5, o_7\}$$
$$E_3 = \{O_3\}$$
$$E_4 = \{o_6\}$$

Let the target set be $O = \{o_3, o_5\}$ then $\underline{O} = \{o_3\}$ and $\overline{O} = (\{o_3\} \cup \{o_2, o_5, o_7\})$ and so $Apr(O) = (\{o_3\}, \{o_3\} \cup \{o_2, o_5, o_7\})$ is a rough set.

Definition 2.4: [5] Let $K = (U, \theta)$ be an approximation space and $\star$ be a binary operation defined on U. A subset $G(\neq \emptyset)$or universe U is called a rough group if $Apr(G) = (\underline{G}, \overline{G})$ satisfies the following property:

(1) $x \star y \in \overline{G}, \quad \forall\ x, y \in G.$
(2) Association property holds in $\overline{G}$.
(3) $\exists, e \in \overline{G}$ such that $x \star e = e \star x = x,\ \forall\, x \in G$; e is called the rough identity element.
(4) $\forall x \in G, \exists\, y \in G$ such that $x \star y = y \star x = e$; y is called the rough inverse element of x in G.

Definition 2.5: [7] Let (U_1, θ) and (U_2, θ) be two approximation spaces, $*$ and $\bar{*}$ be two operations over U_1 and U_2, respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$. $Apr(G_1)$ and $Apr(G_2)$ are called homomorphic rpugh set if there exsts a mapping ϕ of G_1 into G_2 such that

$$\forall\, x, y \in \overline{G}_1, \qquad \phi(x * y) = \phi(x)\ \bar{*}\ \phi(y)$$

If ϕ is 1-1 mapping $Apr(G_1)$ and $Apr(G_2)$ are called isomorphic rough sets.

Definition 2.6: [15] An algebraic system $(Apr(R), +, *)$ is called rough ring if it satisfied:

(1) $(Apr(R), +)$ is a rough commutative addition group.

(2) $(Apr(R), *)$ is a rough multiplicative semi-group.

(3) $(x + y) * z = x * z + y * z$ and $x * (y + z) = x * y + x * z,\ \forall\, x, y, z \in Apr(R).$

Definition 2.7: [25] Let $(Apr(R), +, *)$ be a rough ring with unity, $(Apr(M), +)$ a rough commutative group. $Apr(M)$ is called a rough left module over the ring $Apr(R)$ if there is a mapping $\overline{R} \times \overline{M} \to \overline{M},\ (a, x) \to ax$ such that

(1) $a(x + y) = ax + ay,\ \ a \in Apr(R);\ x, y\ \in Apr(M)$

(2) $(a + b)x = ax + bx;\ \ a, b \in Apr(R); x \in Apr(M)$

(3) $(ab)x = a(bx);\ \ a, b \in Apr(R);\ x \in Apr(M)$

(4) $1x = x$, 1 is a unit element of $Apr(R)$ and $x \in Apr(M)$

A rough right module over the ring $Apr(R)$ can be defined similarly. Who do not require rough ring to be unital omit condition (4).

Definition 2.8: [11] A rough subset $Apr(N) \neq \emptyset$ of a rough module $Apr(M)$ is called rough sub-module of $Apr(M)$, if $Apr(N)$satisfies the following:

(1) $Apr(N)$ is a rough subgroup of $Apr(M)$

(2) $ay \in \overline{N},\ \forall a \in Apr(R)$ and $y \in Apr(N)$.

Definition 2.9: [25] Let $Apr(M)$ and $Apr(M')$ be two rough R-moduel. If there exists a mapping η of M into M' such that

(1) η is a homomorphism of a rough group $Apr(M)$ into $Apr(M')$;

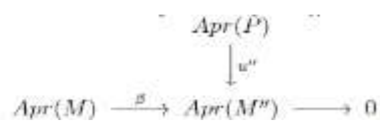
(2) $\eta(ax) = a\eta(x)$, $a \in Apr(R), x \in Ap(M)$

then η is called a homomorphism of rough module $Apr(M)$ into $Apr(M')$. If η is a 1-1 mapping, it is called an isomorphism of rough module $Apr(M)$ into $Apr(M')$.

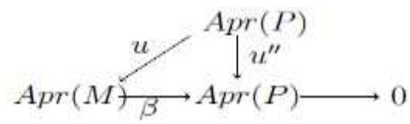
III. Rough Projective Module

Definition 3.1: A sequence $Apr(M') \xrightarrow{\alpha} Apr(M) \xrightarrow{\beta} Apr(M'')$ of two homomorphism of a module over the ring $Apr(R)$ is said to be rough exact if $Im(\alpha) = \ker(\beta)$. This happens if and only if $(i)\beta\alpha = 0$, and (ii) the relation $\beta(x) = 0, x \in Apr, (i.e. x \in \overline{M}\ and\ x \in \underline{M})$, implies that $x = \alpha(x')$ for some $x' \in Apr(M')$. Indeed condition (i) and (ii) mean respectively that $Im(\alpha) \subset \ker(\beta)$ and $\ker(\beta) \subset Im(\alpha)$.

Definition 3.2: An $Apr(R)$-module $Apr(P)$ is projective if and only if every diagram



with exact row (i.e., with β surjective) can be completed to a commutative diagram



by means of a homomorphism $u: Apr(P) \to Apr(M)$. Any homomorphism $u: Apr(P) \to Apr(M)$ for which $u'' = \beta u$ is called a lifting of u''(over β); thus $Apr(P)$is projective if and only if any homomorphism $u''$of $Apr(P)$ into any quotient $Apr(M'')$ of any $Apr(R)$-module $Apr(M)$ can be lifted to a homomorphism u of $Apr(P)$ into $Apr(M)$.

Definition 3.3: Let $Apr(R)$ be a rough ring and let $Apr(M)$ be a $Apr(R)$-rough module. A subset $S \subseteq Apr(M)$is said to be $Apr(R)$-linearly dependent if there exist distinct $x_1, x_2, x_3, \cdots, x_n$ in S and elements $a_1, a_2, a_3, \cdots, a_n$ of $Apr(R)$, not all of which are zero, such that

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

A set that is not $Apr(R)$-linearly dependent is said to be $Apr(R)$-linearly independent.

Definition 3.4: Let $Apr(M)$ be an $Apr(R)$-rough module. A subset S of $Apr(M)$ is a basis of $Apr(M) \neq \{0\}$ if and only if every $x \in Apr(M)$ can be uniquely written as

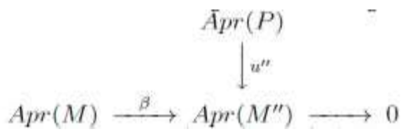
$$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

for $a_1, a_2, \cdots, a_n\ \in Apr(R)$ and $x_1, x_2, \cdots, x_n \in Apr(S)$.

Definition 3.5: An $Apr(R)$-rough module $Apr(M)$ is a rough free module if it has a basis.

Example 3.1: Every rough free module is rough projective module.

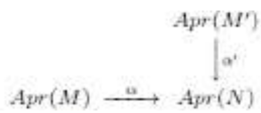
Solution: For, let us be given the diagram



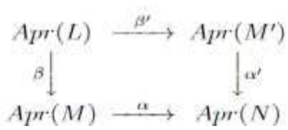
and suppose that $Apr(P)$is rough free module, take a basis $z_i, i \in I$ of $Apr(P)$ and set $x'' = u''(z_i), i \in I$. Since β is surjective, their exist elements $x_i \in Apr(M)$ such that $\beta(x_i) = x_i'', i \in I$. As $Apr(P)$is rough free module with basis $z_i, i \in I$, There exists a (unique) homomorphism $u: Apr(P) \to Apr(M)$ such that $u(z_i) = x_i, i \in I$. Since $\beta\big(u(z_i)\big) = \beta(x_i) = x_i'' = u''(z_i)$ for every $i \in I$, therefore $\beta u = u''$, and the proof of our assertion is complete.

Example 3.2: There are modules that are not rough projective module for example, $\prod_N Z_i$, where $Z_i = Z$ for $i = 1, 2, 3, \cdots$, is not a rough projective Z-module.

Preposition 3.1: Let us be given two coterminal homomorphism on rough projective modules $\alpha: Apr(M) \to Apr(N)$, $\alpha': Apr(M') \to Apr(N)$ and form the diagram



a pullback of α, α' or of the above diagram, is a pair of coinitial mappings $\beta: Apr(L) \to Apr(M)$, $\beta': Apr(L) \to Apr(M')$ such that the square



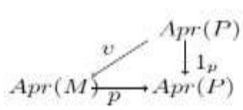
is commutative.

Theorem 3.1: An $Apr(R)$-module $Apr(P)$ is rough projective if and only if every rough exact sequence of the form

$$0 \to Apr(M') \to Apr(M) \xrightarrow{p} Apr(P) \to 0 \qquad (1)$$

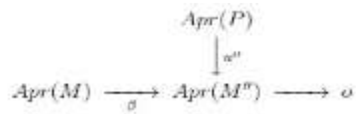
Splits.

Proof: If $Apr(P)$ is projective and (1) a rough exact sequence, then lifting the identity endomorphism 1_p of p to a homomorphism of $Apr(P)$ into $Apr(M)$ we obtain a homomorphism $v: Apr(P) \to Apr(M)$

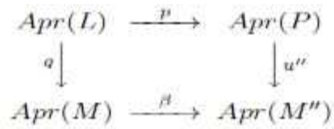


such that $pv = 1_p$. Therefore the sequence (1) splits.

Conversely: suppose that every sequence of the form (1) splits, and let us be given the diagram



with β surjective. Form the pull-back



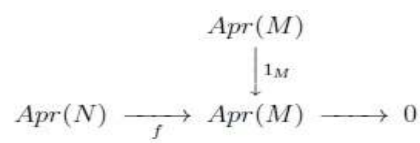
of the above diagram, since β is surjective, so is p ; therefore denoting by $Apr(L')$ the kernel of p, we have the exact sequence

$$0 \to Apr(L') \to Apr(L) \xrightarrow{p} Apr(P) \to 0$$

Since this sequence splits, there exists $v: Apr(P) \to Apr(L)$ such that $pv = 1_p$. Then $u = qv$ is a homomorphism from $Apr(P)$ to $APr(M)$, and have $\beta u = \beta qu = u''pu = u''$. Hence $Apr(P)$ is rough projective.

Preposition 3.2: If $f: Apr(N) \to Apr(M)$ is an epimorphism and $Apr(M)$is a rough projective $Apr(R)$-module, then $Apr(M)$is isomorphic to direct summand of $Apr(N)$.

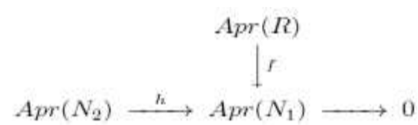
Proof: Since the row exact diagram



can be completed commutatively by an $Apr(R)$-linear mapping $g: Apr(M) \to Apr(N)$such that $fg = id_M, g$ is a splitting map for f and g is a homomorphism and $Apr(N) = Im(g) \bigoplus \ker f$. So the result follows since $Im(g) \cong Apr(M)$.

Lemma 3.1: The rough ring $Apr(R)$ is a rough projective $Apr(R)$-module.

Proof: We need to show that any row exact diagram



can be completed commutatively by an $Apr(R)$-linear mapping $g: Apr(R) \to Apr(N_2)$. If $F(1) = y$ and $x \in Apr(N_2)$ such that $h(x) = y$, let $g: Apr(R) \to Apr(N_2)$ be defined by $g(a) = xa$. Then g is well defined, $Apr(R)$-linear and $f = hg$.

IV. Conclusion and Future Work

Rough set theory is a new powerful Mathematical tool for dealing uncertain problems. Recently, rough set theory has received wide attention in the real life applications and the algebraic studies. In recent years, the combination of rough set theory and abstract algebra has many interesting research topics. In this paper we focused on algebraic results by combining rough set theory and abstract algebra. In other words we have provided an algebraic viewpoint for rough set theory and we hope the results given in this paper can further enrich rough set theories. Naturally applying our results to other fields i.e. applications of module theory, is also a valuable work and we will present it in the future work.

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