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Automatization Parameterized Exponential Generating Functions for Addition Theorem

Sesappa A Rai

Abstract: Generating functions play a very important role in most of the Mathematical and statistical problems. The present study has derived and discussed the addition theorem of special functions whose generating function is exponential. The code of the derivative is also discussed. Exponential generating functions and other special functions have high importance in addition theorem, applicable in various recently developing fields like Computational electro-magnetic fields and other computational fields. It works by automatization in Fast Multi-pole Method using Singular Value Decomposition (SVD). Its code can be applied using Mathematica, Matlab and other software packages.

Keywords - Generating functions, addition theorem, automatization.

1.1 Introduction

A bivariate function g(t, s) in a bilinear form is the starting point for developing acceleration techniques and to develop computational methods for deriving numerical solution of an optimization problem,

$$\max_{x,b} \sum_{t\in T}\sum_{s\in S} b(t)g(t,s)x(s),$$

or, a system of equations

$$\sum_{s\in S} g(t,s)x(s) = b(t), \quad s\in S, \; t\in T,$$

where T and S are large subsets of discrete points R^n, wherein b(t) is defined on *T* and x(s) defined on *S.* Numerical solutions to such large problems, resort in general, to iterative methods, where g(t, s) is evaluated or re-used many times. The sample points in *S* or *T* may not be equally spaced along each dimension of R^n. The sample points may change from an iteration step to the next. Here, g(t, s) may represent the response of b at t upon x at s, or the interaction or correlation between b and x at t and s. We refer to g(t, s) simply as an interaction function.

When an interaction function is provided with large data sets *T* and *S*, we seek the potential in compressive representation for computational efficiency. The previous problems can be described aggregately in their respective matrix forms.

$$G(T,S)x(S) = b(T),$$

A compressed expression of the matrix G(T,S), or its sub-matrices, leads to an efficient algorithm for operations with the matrix. Note that any sub matrix of $G(T',S')$ is associated with a subset $S' \subset S$ and a subset $T' \subset T$ and vice versa. The matrix $G(T',S')$ may be a sub matrix of another matrix.

Department of Mathematics,
Manipal Institute of Technology,
Manipal University,Udupi TQ
Manipal- 576 104, India
www.manipal.edu

1.2 **Expansion-induced compressive representation**

We briefly describe the expansion approach to obtaining a compressive expression of a sub matrix $G(T',S')$. Suppose we have an expansion of g(t, s) for all $(t,s)\in T'\times S'$ as follows:

For any $\varepsilon > 0$, there are natural numbers M and N so that

$$g(t,s) = \sum_{m=0}^{M}\sum_{n=0}^{N} \phi_m(t,t_c).h_{mn}(t_c,s_c).\psi_n(s,s_c) + \in_{MN}(t,s) \qquad (1)$$

$$|\in_{MN}(t,s)| \le \in, \quad \forall s\in S', \; \forall t \in T',$$

with respect to a pair of reference points t_c and s_c. The expansion is bilinear in ϕ_m and ψ_n with the separation of variables t and s, via the reference points. In matrix form,

$$G(T',S') = \phi_M(T', t_c)^T H_{MN}(t_c, s_c)\psi_N(S', s_c) + E_{MN}(T', S'),$$

where the matrices are composed of the corresponding elements in (1). With a translation-invariant interaction function, the reference points in the expansion (1) are translations in **t** and **s**, respectively. In the matrix form,

$$G(T'-S') = \phi_M(T'-t_c)^T H_{MN}(t_c - s_c)\psi_N(S'-s_c) + E_{MN}(T',S') \qquad (2)$$

The first term may render a compressed and sufficiently accurate representation of the sub matrix. A compressed factorization is obtained when the number of non-zero elements in H_{MN} are sufficiently smaller than that of $G(T',S')$. This condition is met when MN are sufficiently smaller than $|T'||S'|$, or when H_{MN} is sufficiently sparser, especially, when H_{MN} is diagonal.

Hitherto, we refer to (2) as a bilinear translative (BiT) factorization. The BiT factorization is recursive if the translation factor $H_{MN}(t-s)$ also has a BiT factorization, and so on. A recursive BiT factorization may lead to a data or matrix compression at multiple levels. The analysis and algorithm development for fast calculation via such recursive translations originate in the Fast Multipole Method [2,6,8]. The methodology has been applied successfully to many computational problems with special interaction functions [3,7, 8].

With the expansion-compression relation in place as described above, one or more than one bilinear translative expansions remains to be found. Some expansions may lead to better compressions than others. However, the derivation of BiT expansions, can be problematic. The exploration of such expansions in exists, or in variety, is limited by individual skills, experience or time. A manual derivation is often tedious and error-prone. Available symbolic computation platforms [3,6,7,8] do not provide direct aid for this particular need.

We address these issues by introducing a calculus approach to BiT expansions for a broad family of functions.

1.3 The Calculus for BiT expansions

The calculus approach starts with bilinear expansions of a particular expansion form,

$$g(x,t)=\sum_{k=L}^{U}\gamma_k(x)\phi_k(t),$$

where $\phi_k(t)$ are linearly independent functions over some open region of **t**. We refer to $\phi_k(t)$ as basic functions, or expansion terms. We use initially the integer power function $\phi_k(t)=t^k$ as the expansion term. We refer to $\gamma_k(x)$ as the coefficient functions. The index ranges between the lower bound ***L*** and the upper bound U, $L\le U$. Each of the bounds may be finite or infinite. The series expansion is semi-bounded when either ***L*** or ***U*** is finite; it is bounded when both ***L*** and ***U*** are finite. A coefficient function is zero when its index is outside of the bounds. Two features of the expansion shall be noticed immediately. First, the variable **x** and **t** are separated in the bilinear expansion form. Second, the function g(x, t) may be viewed as the generating function that defines or encodes coefficient functions $\gamma_k(x)$.

The calculus for bilinear expansion introduced here is distinguished in three main aspects. (A) The functions are not restricted to those with the zero lower bound expansion. More specifically, expansions with pole terms, [1,5,10] i.e., negative power terms in **t,** are included. (B) Under certain conditions, the roles of term functions and coefficient functions in the bilinear representation can be exchanged. (C) The calculus is compact in itself. It consists of a small set of elementary expansion rules and a small set of initial expansion cases. The approach is illustrated with numerous function expansions that have special positions in both mathematical analysis and numerical computation.

1.4 Elementary Expansion Rules

In this section the power function t^k is used as the basis functions,

$$g(x,t)=\sum_{k=L}^{U}\gamma_k(x)t^k. \qquad (3)$$

For convenience in expression, from time to time the equivalent form

$$g(x,t)=\sum_{k=L}^{U}\sigma_k(x)t^k/k!.$$ is used.

Variable **t** is associated with the basis functions, and variable **x** is associated with the coefficient, or generated, functions. Note two special cases

. In the first case, L = U = 0, g(x, t) = g(x). In the second, at any specific value x_*, $g(x_*,t)$ corresponds to a scalar sequence $\{\gamma_k(x_*)\}$.

The base cases

For bounded series, start with simplest polynomials, the constant 1 and the linear functions **x** and **t**. An elementary semi-bounded expansion is as follows,

$$e^{xt}=\sum_{k=0}^{\infty}\frac{1}{k!_k}x^k t^k. \qquad (4)$$

The variables **x** and **t** are separated naturally in the expansion, by the Euler's identity $\exp(it)=\cos t+\sin t$, where

$$\mathrm{Cos}(xt)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}t^{2k},$$

$$\mathrm{Sin}(xt)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}t^{2k+1}.$$

2.1 Bilinear expansion of power functions

Example 1. The binomial expansion,. a direct application of the rule Hadamard product to the right-hand side of the identity $e^{x+y}=e^x.e^y$ recovers the familiar binomial expansion of $(x+y)^k$ for $k\ge 0$ the following symmetric representation is used,

$$\frac{(x+y)^k}{k!}=\sum_{p=0}^{k}\frac{y^{k-p}}{(k-p)!}\frac{x^p}{p!},\quad k\ge 0. \qquad (5)$$

This expansion contributes to the establishment of Theorem 1.The next two examples illustrate the use of the division rule. They show the essential difference among bilinear expansions of $(x+y)^k$ between the case of positive powers and the case of negative ones in **t.** The latter is not bounded in bilinear expansions and imposes more condition on the expansion.

Example 2. The bilinear expansion of $\frac{1}{c+ax+by}$. This includes in particular, the reciprocal difference,

$$\frac{1}{y-x}=\sum_{p=0}^{\infty}\frac{x^p}{y^{p+1}}. \qquad (6)$$

The expansion holds when $|y|>|x|$. The function $\frac{1}{y-x}$ is also named the Cauchy function. It is the kernel function of the Cauchy integral formula and for the Hilbert transform. This identity is related to the well-known Neumann's summation. Here, it is observed that (6) may be also seen as the expansion of powers of y.

Example 3**.** The expansion of the reciprocal of a quadratic polynomial

By the division rule,

$$\begin{aligned}&\frac{1}{c+ax^2+bxt+t^2}=\sum_{k=0}^{\infty}\alpha_k(x)t^k,\qquad c+ax^2\neq 0,\\&\alpha_0(x)=(ax^2+c)^{-1},\qquad \alpha_1(x)=-bx\alpha_0(x)^2\\&\alpha_{k+1}(x)=-\alpha_0(x)(bx\alpha_k(x)+\alpha_{k-1}(x)),\end{aligned} \qquad (7)$$

For a special instance,

$$\frac{1}{(y-x)^2}=\sum_{k=0}^{\infty}\frac{k+1}{y^{k+2}}x^k. \qquad (8)$$

This can be also obtained by squaring the expansion of the Cauchy function in (6)

2.2 Recursive Translations

Thus bilinear expansion rules for elementary operations have been developed. Now, the bilinear translative (BiT) expansion is to be developed especially with recursive translations. Many functions of interest, such as distance related functions, are translation invariant, namely $g(t,s)=g(t-h,\ s-h)$ for arbitrary translation h. In particular, at $h=s, g(t,s)=g(t-s)$.

The Cauchy function and the reciprocal distance function, discussed earlier, are translation invariant. Among others, the translated Gaussian $\exp(x-t)^2/\sigma)$, $\sigma>0$, appears in a wide range of application problems. When a translation invariant function is expanded in the form (3), a bilinear and translative (BiT) expansion is readily available.

3.1 The recursive translation theorem

In this section the rule of translation and illustration for the application of the rule is introduced.

Theorem 1: Assume $g(x,y)$ is translation invariant and semi-bounded in expansion,

$$g(x-t)=\sum_{k=0}^{U}\gamma_k(x)t^k/k!.$$

Then,

$$g(x-y)=\sum_{p=0}^{U}\sum_{q=0}^{U-k}\frac{(x-x_c)^p}{p!}\gamma_{p+q}(x_c-y_c)\frac{(y_c-y)^q}{q!}. \qquad (9)$$

where the reference points x_c are chosen according to the expansion condition. Furthermore,

$$\gamma_k(x-y)=\sum_{p=0}^{U-k}\gamma_{k+p}(x)\frac{y^p}{p!}, \qquad 0\le k\le U. \qquad (10)$$

Proof: Since $x-y=(x_c-y_c)-[(y-y_c)-(x-x_c)]$, the expansion of g is,

$$g(x-y)=\sum_k\gamma_k(x_c-y_c)\frac{[(y-y_c)-(x-x_c)]^k}{k!}, \quad 0\le k\le U.$$

This leads to the BiT expansion in (9) with the binomial expansion of $[(y-y_c)-(x-x_c)]^k$. Next,

$$g(d-t)=\sum_{k=0}^{U}\gamma_k(d)\frac{t^k}{k!}=\sum_{p=0}^{U}\gamma_p(x+d)\frac{(x+t)^p}{p!}$$

$$=\sum_{p=0}^{U}\sum_{k=0}^{p}\gamma_p(x+d)\frac{x^{p-k}}{(p-k)!}\frac{t^k}{k!}=\sum_{k=0}^{U}\sum_{p=k}^{U}\gamma_p(x+d)\frac{x^{p-k}}{(p-k)!}\frac{t^k}{k!}.$$

By comparison, in terms of t^k, we have

$$\gamma_k(d)=\sum_{q=0}^{U-k}\gamma_{k+q}(x+d)\frac{x^q}{q!}.$$

Let y = d + x,then we get(10).

The recursive feature of (9) lies in the self-referenced bilinear expansion of $\gamma_k(x-y)$ in(10). Apply (10) twice, we get

$$\gamma_p(x-y)=\sum_{p=0}^{U-k}\sum_{q=0}^{U-2k}\frac{(x-x_c)^p}{p!}\frac{t^k}{k!}\gamma_{k+p+q}(x_c-y_c)\frac{(y_c-y)^q}{q!}.$$

For a simple example, the theorem can be applied directly to (8) for a BiT expansion of the Cauchy function with coefficient functions $\gamma_k(y)=1/y^{k+1}$, $k\ge 0$.

3.2 A BiT expansion of the Gaussian

The translated Gaussian $g(t,s)=\exp(-(t-s)^2/\sigma^2)$ appears in a wide range of computational applications [10]. A bilinear expansion is first developed. With normalization, it is assumed $\sigma=1$. Factorization of the translated Gaussian, $g(t,s)=e^{-s^2}e^{2st}e^{-t^2}$ and the expansion of $\exp(-t^2)$ from the base case can be done by applying the rule of substitution of $x=(-1)$ and the rule of dilation with $d=2$. By the rule of Hadamard product and the rule of diagonal product, this would result in

$$e^{2st-t^2}=\sum_{k=0}^{\infty}H_k(s)\frac{t^k}{k!},\ e^{-(t-s)^2}=\sum_{k=0}^{\infty}h_k(s)\frac{t^k}{k!}$$

with

$$H_k(s)=k!\sum_{q=0}^{\lfloor k/2\rfloor}\frac{(-1)^q(2s)^{k-2q}}{(k-2q)!q!},\ h_k(s)=e^{-s^2}H_k(s).$$

(The Hermite function is given by [10]), where $\lfloor . \rfloor$ is the floor function. In fact, $H_k(s)$ and $h_k(s)$ are known as the Hermite polynomials and the associate Hermite functions, respectively. That is, $\exp(-(t-s)^2)$ is the generating function of the Hermite functions.

The bilinear expansion of the translated Gaussian satisfies the condition of Theorem 1. Thus,

$$e^{-(t-s)^2}=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t-t_c)^p}{p!}h_{p+q}(t_c-s_c)\frac{(s_c-s)^q}{q!}.$$

And the Hermite function $h_k(t-s)$ has the bilinear expansion

$$h_k(t-s)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(t-t_c)^p}{p!}h_{p+q+k}(t_c-s_c)\frac{(s_c-s)^q}{q!}.$$

4. Logarithm

In this section the expansion rules for the most common compositions among bivariate functions is introduced. First the special rule for integer powers and root extraction, then the rule for logarithm and finally the anti-logarithm operation with variable roots and powers is introduced.

To describe the expansion rule for logarithm, it suffices to use the natural base.

Theorem 2**:** Assume that a(x,t) is positive and semi-bounded in expansion

$a(x,t)=\sum_{k=0}^{\infty}\alpha_k(x)t^k$, with $\alpha_0(x)\ne 0$, Then, the coefficient functions of $g(x,t)=\log(a(x,t))$ in the expansion (3) are as follows:

$$\gamma_0(x)=\log(\alpha_0(x)),\gamma_1(x)=\frac{\alpha_1(x)}{\alpha_0(x)},$$

$$\gamma_{k+1}(x)=\frac{\alpha_{k+1}(x)}{\alpha_0(x)}-\frac{1}{(k+1)\alpha_0(x)}\sum_{p=0}^{k-1}(p+1)\gamma_{p+1}(x)\alpha_{k-p}(x).$$

This rule remains valid with the term reversal.

Proof: Assume the conditions,

$\gamma_0(x)=g(x,0)=1na(x,0)=1n\alpha_0(x)$. and consider ∂/dt on both sides of $g(x,t)=\log(a(x,t))$, which leads to $g_t.a=a_t$, where $g_t=\partial g/\partial t$. The recursion would follow term by term matching. In the case that the finite lower or upper bound ℓ is non-zero, the application of the logarithm rule is preceded by a shift of ℓ terms and followed by an addition of $\ell\ln(t)$, for t>0,

$$\log(t) = 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\left(\frac{t-1}{t+1}\right)^{2k+1}, \quad t>0.$$

5. Generation of the associated Laguerre polynomials

The use of expansion rule for variable powers with function is illustrated,

$$g(x,t) = \frac{e^{-xt/(1-t)}}{(1-t)^{\ell+1}}, \qquad (11)$$

with a natural number ℓ.This function is known as generating function for Laguerre polynomials with $\ell = 0$ and the associated Laguerre polynomials $L_k^{\ell}(x)$ with $\ell > 0$. These polynomials with expansion rules from simple initial cases are generated without resort to additional resource. Consider first the expansion of $r(x,t) = b(x,t)^{a(x,t)}$ with $b(x,t) = e^{-xt}$ and $a(x,t) = (1-t)^{-1}$.

Denote by $\rho_k(x)$ the coefficient functions of $(1-t)^2 r_t(x,t) = -xr(x,t)$. is obtained.

It follows ,from this equation,

$$\rho_0(x) = 1, \ \ \rho_1(x) = -x,$$

$$\rho_{k+1}(x) = \frac{1}{k+1}[(2k-x)\rho_k(x) - (k-1)\rho_{k-1}(x)], \ k>1.$$

Finally, the division rule is applied to $r(x,t)/(1-t)^{\ell+1}$ to obtain the coefficient functions for $g(x,t)$,

$$L_k^{\ell}(x) = \rho_k(x) - \sum_{p=1}^{k}(-1)^p C_{\ell+1}^p L_{k-p}^{\ell}(x),$$

where $C_{\ell+1}^p$ are the binomial coefficients. In particular the first couple of associated Laguerre polynomials for each $\ell \geq 0$. is listed.

$$L_0^{\ell}(x) = 1, \ L_1^{\ell}(x) = 1+\ell-x, \ L_2^{\ell}(x) = \frac{(\ell+2)((\ell+1-2x)+x^2}{2}$$

Details of the simplification are skipped.

6. 1 Addition theorem

Binomial expansion for power functions of addition theorem are extended for the functions ϕ_k, defined by a generating function $g(x,t)$ of the form (7).

Theorem:3 Assume that functions $\gamma_k(x)$, $\alpha_k(x)$ and $\beta_k(x)$ are generated by g(x, t), a(x, t), and $b(x,t)$, respectively, with the same lower bound L on k, L = 0 or $L = -\infty$ assuming further that

$$g(x+y,t) = a(\mu_a x, \lambda_a t) b(\mu_b y, \lambda_b t), \qquad (12)$$

For some scalars λ_* and μ_* Then $\gamma_k(x)$ has the addition expansion,

$$\gamma_k(x+y,t) = \sum_{p=L}^{k-L}\lambda_a^p \alpha_p(\mu_a x)\lambda_b^{k-p}\beta_{k-p}(\mu_c y), \qquad (13)$$

The addition expansion in (12) is bounded when L = 0 and unbounded when $L = -\infty$. The proof is based on the condition of (12) and application of the multiplication rule.

If g = a = b, $\gamma_k(x)$ are closed among themselves in the expansion of addition, in the variable x. If g, a and b belong to a family of functions, $\gamma_k(x)$ are closed under addition within the family.

A couple of function families discussed this far are closed under the addition expansion. They include the integer power functions themselves by the generating function e^{xt} the exponential functions $\exp(kx)$ by $\exp(e^{ixt})$, and the Hermite polynomial by $\exp(2xt - t^2)$, [1]. In the latter case one finds that $\lambda = 1/\sqrt{2}, \mu = \sqrt{2}$ and L=0. Thus, addition theorem for the Hermite polynomials is as follows:

$$\frac{H_k(x+y)}{k!} = \frac{1}{2^{k/2}}\sum_{p=0}^{k}\frac{H_p(\sqrt{2}x)}{p!}\frac{H_{k-p}(\sqrt{2}y)}{(k-p)!};$$

or

$$H_k(x+y) = \frac{1}{2^{k/2}}\sum_{p=0}^{k}\binom{k}{p}H_p(\sqrt{2}x)H_{k-p}(\sqrt{2}y)$$

Defining $g_\ell(x,t) = e^{-xt/(1-t)\ell+1}$ for $\ell \geq -1$. For each ℓ, $g_\ell(x,t)$ generates a family of polynomials $L_k^{\ell}(x), \ k \geq 0$. This quickly verifies that $g_\ell(x+y) = g_n(x)g_{\ell-n-1}(y)$ for any n, $-1 \leq n \leq \ell$

Thus, $L_k^{\ell}(x+y) = \sum_{p=0}^{k} L_p^n(x) L_{k-p}^{\ell-n-1}(y).$

That is $L_k^{\ell}(x), \ k \geq 0$ are closed under addition, and the others are closed under addition in the family for all $\ell \geq -1$. From the associated Laguerre polynomials, it is observed that expansion of the function family is for convenience in addition expansions. In fact, another addition expansion is got for the Hermite polynomials, using the decomposition $e^{2(x+y)t-t^2} = e^{2xt-t^2}e^{2yt}$ .The first factor on the right hand side generates $\frac{H_k(x)}{k!}$, and the second generates $\frac{(2y)^k}{k!}$.

6.2 Expansions in the Bessel Functions

In this section, expansion of the Bessel function for integer order, $J_k(x)$, is illustrated, it is seen that the generating function is $\exp(x(t-1/t)/2)$. Bessel functions of integer order are important in their own right. Certain special properties are first described and then BiT expansion for Bessel functions are discussed.

7.2 Special properties

The properties directly relevant to BiT expansion of interest are described. Let $g(x,t) = \exp(x(t-1/t)/2)$. The generation function satisfies by condition and in particular, the addition function for Bessel function is as follows, $J_k(x+y) = \sum_{p=-\infty}^{\infty} J_p(x)J_{k-p}(y).$

That is, these functions are closed under addition among themselves.

The Bessel functions are symmetric about the origin, namely, $J_k(-x) = (-1)^k J_k(x)$. This can be verified easily by matching the terms t^k on both sides of the identity $g(-x,t) = g(x,t)$. The integer power functions have the same properties about and at the origin..

The condition in Theorem 3 for the recurrence in BiT expansion requires that the expansion in $J_k(x)$ be unique. In fact, any function expanded in terms of the Bessel functions can be uniquely represented in terms of the Bessel functions of natural order. The generating function is invariant under the term reversal and negation, i.e., $g(-x,t) = g(x,-1/t)$. Therefore,

$$J_{-k}(x) = (-1)^k J_k(x). \qquad (14)$$

Thus, the conversion to expansion of Bessel functions of natural order is straightforward. Therefore the following expansion format is used

$$. \; f(x,y) = \phi_0(y) J_0(x) + 2\sum_{k=1}^{\infty} \phi_k(y) J_k(x). \qquad (15)$$

Further the variable **x** is associated with basis functions, indicating a hierarchy relationship. The basis functions are defined by a generating function in terms of t^k. In particular, the generating function is turned the other way around,

$$g(x,t) = e^{x(t-1/t)/2} = J_0(x) + \sum_{k=1}^{\infty} (t^k + (-1/t\ )^k) J_k(x) \quad (16)$$

The expansion of this function in the Bessel basis is semi-bounded while the expansion in the integer power functions is unbounded.

The expansion transformation (16) is the result of base transformation and integral representation of the coefficients of a function in Bessel basis. However, it no longer involves the basis transformation or integral calculation. The derivation is also based on Hadamard product rule, which permits the pole. Note that there is a recursion relation among the coefficients of Neumann's polynomials.

Conclusion

The basic theory and calculus for bilinear expressions of bivariate functions are presented, especially for recursive bilinear expansions and translation-invariant functions. There are only five basic rules for bilinear expansions, namely, the rules i) variable substitution, ii) linear combination iii) multiplication, iv) logarithm and v) antilogarithm that includes division. With these rules, expansions of various functions can be reached from only a few initial cases, Additional three rules exist for recursive BiT expansions, vi) and addition theorem.

Some well-known expansions are recovered as illustrative and manifesting examples. They include the generation of Hermite, Legendre, Laguerre polynomials as well as their addition theorems. Other expansions in our illustration are rarely seen in popular literature, such as transformation from the natural power functions to the natural order Bessel functions, or the expansion of the Cauchy function in the Bessel basis. The theory and the calculus method has not only unified these expansions, but also simplified or demystified the derivation process. The calculus renders bilinear expansions and recursive BiT expansions as a computation procedure in itself.

Although most of the examples are one dimensional in the source and target coordinates, the expansion method can be applied to high dimensional problems in a Cartesian coordinate system, one dimension at a time. In fact, the expansions in Bessel functions can be also carried over straight forwardly to two-dimensional problems in cylindrical coordinate system.

The generalized expansion theory and calculus has extended the scope of recursive BiT expansions for translation-invariant functions. This generalization also offers the variety in expansion for exploring potential in compression or numerical property in finite-precision computations, The impact of the generalized theory is on the selection of, or the transformation to, an adequate basis for compressive representation for any given situation. To this very purpose, this computation platform supports expansion calculus which is much needed to facilitate and accelerate the process of algorithm development.

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