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Geometrical approach to the approximation of the volume of a solid of revolution, and comparative analysis with existing methods

[Sharan M Rai]

***Abstract* - Solids of revolution have applications in various fields such as manufacturing (casting, machining), computer aided designing etc., wherein axis-symmetric solids are generated by revolution of curves. Most often the curves corresponding to these solids are irregularly shaped. Thus, regular integration cannot be applied to obtain definite volume of such solids. Hence, given a set of function values (radii), an approximation of volume of the solid can be made. The approximation introduced here (frustum approximation), divides the solid into a number of frustums of cones, instead of following the conventional approach of dividing the solid into cylinders. The summation of all the individual volumes of the frustums, gives the approximate volume of the total solid. The study also compares and analyzes the frustum approximation with existing methods of approximating integrals. The results indicate that for all sub-intervals of solids that are 'concave' in nature, the frustum approach generates a better approximation compared to existing methods.**

***Keywords* – Solid of revolution, Approximation of volume of revolution, Frustum Approximation, Trapezoidal Rule**

I. Introduction

A solid of revolution is defined as the solid figure obtained upon rotating a plane region (enclosed by a function f(x), its limits, and an axis) around an axis. The concept of solids of revolution is widely used in engineering, especially in the fields of manufacturing engineering, where production of axis-symmetric objects is involved. While constructing any solid having axis-symmetry, it is important to be able to obtain the volume of the solid.

Conventionally, the formula for finding the volume of revolution is derived by dividing the solid into disks of radii $f(x)$, with width dx, and summing them up. Integrating these disks over an interval, will give the exact volume of the solid of revolution.

Sharan Rai
Dept. of Mechanical Engg.
National Institute of Technology, Surathkal
Karnataka, India

$$V = \int_a^b \pi f(x)^2 dx$$

Often, the functions corresponding to the solid, cannot be easily integrated, and in many cases cannot be defined. Thus an approximation that provides the least error for finding the volume of revolution is required.

This study elaborates a geometrical approach to the approximation of the volume of revolution, by dividing the solid into a number of frustums, instead of cylinders. The results obtained are compared with results obtained from the modified form of the trapezoidal method for approximation of integrals.

II. Method of Derivation

Total volume of a solid of revolution, approximation each sub-interval of the solid as a frustum of a cone, can be derived as follows:

Consider a continuous function $f(x)$ defined on $[a, b]$. The region under $f(x)$ is split into n finite equally spaced intervals, having $n + 1$ grid points, i.e. $a = x_0, x_1, x_2, \dots, x_n = b$. Thus region under $f(x)$ is approximated to be the sum of n trapeziums of height $f(x_k)$ and $f(x_{k+1})$, and width $h = \frac{(b-a)}{n}$.

Consider a single interval as shown in Fig.1.Produce the slant side of the trapezium, to a point on the axis. Let the angle made by the slant on the axis be θ. Two similar cones with (radius, height), (r1, h1) and (r2, h2) are considered, whose difference forms the volume of

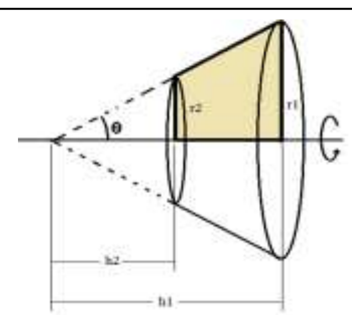

Fig. 1

the frustum. Therefore volume of each frustum V_k is given by,

$$V_k = \frac{1}{3}\pi r_1^2 h_1 - \frac{1}{3}\pi r_2^2 h_2$$

But,

$$\tan\theta = \frac{r_1}{h_1} = \frac{r_2}{h_2}$$

Thus,

$$V_k = \frac{1}{3}\pi \tan\theta^2 \,(h_1^3 - h_2^3)$$

$$V_k = \frac{1}{3}\pi \tan\theta^2 \,(h_1 - h_2)(h_1^2 + h_2^2 + h_1 h_2)$$

$$V_k = \frac{1}{3}\pi \tan\theta^2 \,(h)\left(\frac{r_1^2}{\tan\theta^2} + \frac{r_2^2}{\tan\theta^2} + \frac{r_1 r_2}{\tan\theta^2}\right)$$

$$V_k = \frac{1}{3}\pi(h)(r_1^2 + r_2^2 + r_1 r_2)$$

Therefore the volume of the solid of revolution obtained by the summation of volumes of all individual frustums (assuming equal sub-interval width *h*), is approximated to be,

$$V \approx \frac{1}{3}\pi\left(\frac{b-a}{n}\right)\sum_{k=0}^{n-1}(f(x_k)^2 + f(x_{k+1})^2 + f(x_k)f(x_{k+1})) \qquad (1)$$

This approximation will be referred to as the Frustum Approach or Frustum Approximation, and is applicable for any number of sub-intervals, and requires a minimum of two points, to find the approximation.

III. Results

The frustum approximation must be compared with an approximation which corresponds to it. From the condition of n=1 on the Newton-Cote's quadrature formula, we obtain the *trapezoidal approximation* of area under a function. This formula can be modified to find volume of revolution V, i.e.

$$V \approx \frac{1}{2}\pi\left(\frac{b-a}{n}\right)\sum_{k=0}^{n}[f(x_k)^2 + f(x_{k+1})^2] \qquad (2)$$

The trapezoidal approximation, is applicable for any number of intervals, and requires a minimum of two distinct points for approximation. Thus this approximation aptly corresponds to the frustum approximation.

We compare the given approximation, with various continuous functions, defined on certain intervals. Minimum number of sub-intervals are considered, so as to increase the magnitude of error. The percentage error generated by each approximation is calculated and shown in Table 1.

Table 1

In the above examples, the first five functions show higher error in the trapezoidal approach as compared to the frustum approach. However, the next three functions show more error in the frustum approach.

Function	Frustum approximation % Error	Trapezoidal approximation % Error
$f(x) = x$ $[0,5], 5\ intervals$	0	0.5
$f(x) = 1/x$ $[1,5], 20\ intervals$	0.5467	0.8201
$f(x) = x^3$ $[0,5], 20\ intervals$	0.35	0.8742
$f(x) = e^x$ $[0,5], 20\ intervals$	1.0384	2.0747
$f(x) = \tan x$ $[-\pi/4, \pi/4], 20\ intervals$	0.3179	0.9561
$f(x) = \sqrt{25 - x^2}$ $[0,5], 20\ intervals$	0.2391	0.0625
$f(x) = \sqrt{x}$ $[0,5], 10\ intervals$	0.5244	0
$f(x) = \sin x$ $[0,\pi], 20\ intervals$	0.4103	0
$f(x) = \ln(x)$ $[1,e], 20\ intervals$	0.0452	[illegible]

Considering the above data, an observation can be made as to the relation between the function and the type of approximation suited for the corresponding solid of revolution. By general observation of the volumes generated, it can be stated that the frustum approach yields a better approximation for functions whose volume of revolution appears contracting in nature, while the trapezoidal approach yields a better approximation with bulging volumes of revolution. But, the last function ($f(x) = \ln x$; $x \in [1, e]$; 20 sub-intervals, Fig. 2), is an anomaly, where the volume of revolution is bulging, yet the frustum approximation shows a better approximation. Further analysis can be done using an area distribution graph.

Consider the case, where, given two points (say, (0,0) and (1,1)), the approximate volume of revolution is found using the two approaches. Plotting the area distribution along x, we obtain the graph, given in Fig.3. The area under the curves represent the volume of revolution. Depending upon which of the two curves the actual area distribution tends to, the better approximation for volume may be decided.

The approximation given by the frustum approach, considers a line connecting the two points, and revolves it forming a frustum (represented as a quadratic curve in the area distribution graph). This implies, that for any continuous curve lying entirely below the line joining any two points, the area distribution of the actual volume of revolution, would be a curve lying below the corresponding curve for the frustum approximation. Thus the frustum approach would provide a better approximation for every concave volume of revolution.

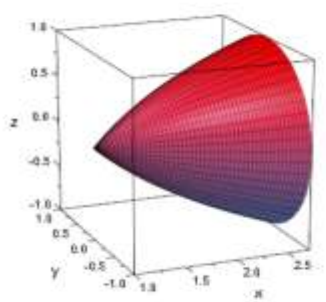

Fig. 2 : Solid of revolution generated by ln(x), x=[1,e]

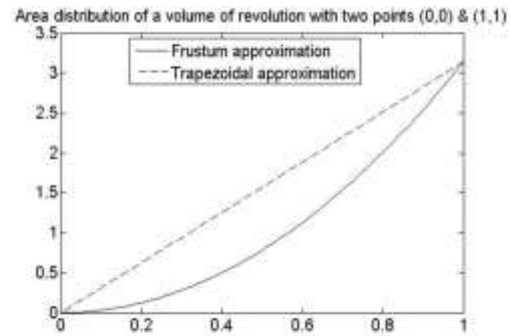

Fig. 3: Area distribution plot of Frustum approximation as well as Trapezoidal

The graph also explains why the trapezoidal approximation provides more error for the volume of revolution of $\ln x$. If the actual area distribution curve lies between the area distribution curves of frustum and trapezoidal approximations, the better approximation cannot be decided until further analysis is carried out. Thus, a generalized statement cannot be formed as to the type of approximation that is apt for convex volumes of revolution.

IV. Conclusion

Given the results from the analysis of the area distribution graph, a hypothesis can be stated:

The frustum approach generates lesser error, as compared to the trapezoidal approach, for functions whose solid of revolution is found to be concave (contracting) or linearly tapering in nature.

These approximations may also be used in the case of complex continuous functions, and the same hypothesis can be modified as follows:

If the second derivative of the modulus of a continuous function in a given sub-interval, is greater than or equal to zero, the frustum approximation generates lesser error, as compared to the trapezoidal approximation, in finding the volume of the solid of revolution in that interval.

The approach introduced in this study can be viewed in another perspective, wherein the curve is first approximated as a number of trapeziums, and then revolved to get frustums. In other words, the trapezoidal approximation was applied to the curve before revolving, and thus generated the frustum approximation.

Although the methods compared in this study give an approximation with greater error as compared to approximations generated by equations of higher orders of the Newton Cotes quadrature formula, (such as Simpsons 1/3 rule (quadratic), which gives significantly less error), a similar approach can be considered in order to obtain possibly a better approximation to the existing Simpsons 1/3 rule. This has to be further analyzed.

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VI. Appendix

Definition of Trapezoidal Rule: In numerical analysis, the trapezium rule (trapezoidal rule) is considered as a linear approach to approximate area under definite integrals, and is derived by applying n=1 condition to the Newton-Cotes quadrature formula. The trapezoidal rule approximates the region under the curve, in a given interval, as a trapezium (or as a summation of trapeziums).

$$\int_a^b f(x)dx \approx \frac{h}{2}\sum_{k=0}^{n}\left(f(x_{k+1}) + f(x_k)\right) = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

Where $h = \frac{b-a}{n}$ is the width of each interval, assuming n equal intervals.

If we consider a function $g(x)$, such that $g(x) = \pi * f(x)^2$, then the above formula would approximate the volume of revolution of a function $f(x)$ about the x-axis.

$$V = \int_a^b g(x)dx = \int_a^b \pi f(x)^2 dx \approx \frac{\pi * h}{2}\sum_{k=0}^{n}(f(x_k)^2 + f(x_{k+1})^2) = \frac{\pi * h}{2}[({y_0}^2 + {y_n}^2) + 2({y_1}^2 + {y_2}^2 + \cdots + {y_{n-1}}^2)] \quad (2)$$

References

[1] Grewal, Bhajan S., and Grewal J. S. "Numerical Differentiation and Integration." *Numerical Methods in Engineering and Science*. 5th ed. Delhi: Khanna, 1999. N. pag. Print.

[2] Atkinson, Kendall E. *An Introduction to Numerical Analysis*. 2nd ed. New York: Wiley, 1989. Print.

[3] Aumann, Günter. "Two algorithms for volume-preserving approximations of surfaces of revolution." Computer-Aided Design 24.12 (1992): 651-657.

[4] Weisstein, Eric W. "Solid of Revolution." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/SolidofRevolution.html

[5] Rahman, Qazi I., and Gerhard Schmeisser. "Characterization of the Speed of Convergence of the Trapezoidal Rule." Numerische Mathematik 57.1 (1990): 123-38. Web.

[6] Weisstein, Eric W. "Newton-Cotes Formulas." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/Newton-CotesFormulas.html

[7] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 885, 1972.

[8] Whittaker, E. T. and Robinson, G. "The Trapezoidal and Parabolic Rules." The Calculus of Observations: A Treatise on Numerical Mathematics, 4th ed. New York: Dover, pp. 156-158, 1967.

[9] M. Abramowitz and I. A. Stegun, eds. Handbook of Mathematical Functions with Formulae, Graphs, and Mathematical Tables. New York: Dover, 1972. (See Section 25.4.)

[10] Atkinson, Kendall E. (1989), An Introduction to Numerical Analysis (2nd ed.), New York: John Wiley & Sons, ISBN 978-0-471-50023-0.

The author is currently pursuing his undergraduate course in the Dept. of Mechanical Engineering, at National Institute of Technology, Surathkal, Karnataka, India

About The Author: