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Applications of the Relative Neighbourhood Graph

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***Abstract*—The relative neighborhood graph of a collection of objects assigns an edge to a pair of objects (A, B), provided that no other object is closer to both A and B, than A and B are to each other. This graph was originally proposed for the purpose of extracting the low-level visual perceptual structure of two-dimensional dot patterns. During the past thirty-four years the relative neighborhood graph has been applied to a multiplicity of different disciplines, and sometimes to several problems within a single discipline. This paper provides a review of some of these applications, including: wireless network communications, archaeological network analysis, grid typification in cartography, data mining for geographic information systems, shape analysis, image morphology, polygon decomposition, the extraction of primal sketches in computer vision, the reduction of the size of the training set in instance-based machine learning, the design of non-parametric decision rules, support-vector machines, cluster analysis, manifold learning, the design of nonparametric tests of the independence of dissimilarity matrices, the design of data-depth measures, testing class separability, estimating two-dimensional voids in the cold dark matter universe, multidimensional data-base indexing, image retrieval, adaptive grid generation for solving partial differential equations, clinical case retrieval in health-care systems, modeling road networks in transportation science, modeling leaf venation patterns in biology, plasmodium machines, swarm intelligence, distributed motion coordination, visualizing metabolic reactions in chemistry, tracking defects in crystal structures, and developing visualization tools such as topological zooming as well as Tukey *scagnostics*.**

***Keywords*—relative neighbourhood graph, Gabriel graph, minimum spanning tree, Urkuhart graph, Delaunay triangulation, proximity graphs, graph theory, computational geometry, image morphology, primal sketch, visual perception, shape analysis, artificial intelligence, support vector machines, machine learning, statistics, wireless networks.**

I. Introduction

At the International Conference on Pattern Recognition in Oxford University in 1979, I presented a graph I named the *relative neighbourhood graph* (RNG), and its application to the extraction of a visual *primal sketch* of a dot pattern [73]. In 1980 I published a paper on this graph in the journal *Pattern Recognition* [70]. Since then the graph has been applied to many problems in a variety of disciplines. The ramifications of the RNG in the 1980s were treated in [37]. Here I briefly revisit these applications and focus on more recent results.

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Two objects are *relative neighbours* if they are at least as close to each other as they are to any other object. This definition presupposes the existsnce of a measure of closeness (similarity) that depends on the specific application. Given a finite collection of objects, the *relative neighbourhood graph* (RNG) is defined as the graph $G(V, E)$, in which V denotes the set of n objects and E the set of m edges, such that there exists an edge e_{ij} in E that connects two objects v_i and v_j in V, if and only if v_i and v_j are relative neighbours. More formally, graph $G(V, E)$ is an RNG provided there exists an edge e_{ij} of weight d_{ij} between v_i and v_j iff $d_{ij} \leq \max(d_{ik}, d_{jk}), \forall\ k \neq i, j, i \neq j$. For a more concrete geometric description let the objects be points in the plane, let the dissimilarity be defined as the Euclidean distance between the points, and refer to Fig. 1. For the two points v_i and v_j construct the circles with centers at v_i and v_j and radius equal to the distance d_{ij} between v_i and v_j (left). The intersection of the two circles (right) determines a region of proximity between v_i and v_j called a *lens* (sometimes referred to as a *lune* in computational geometry). Two points v_i and v_j are at least as close to each other as they are to any other point in a set, if and only if, the lens determined by v_i and v_j does not contain any other point of the set. Hence to construct the RNG of a set of points, two points are joined with an edge if, and only if, their lens is empty.

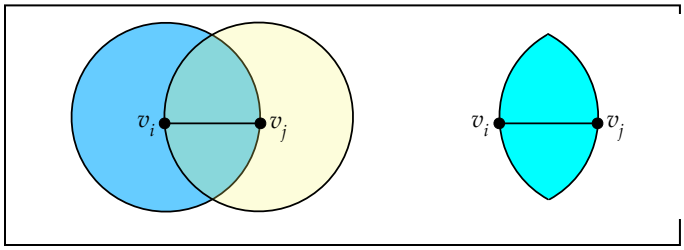

Figure 1. Geometric definition of *relative neighbourhood graph*.

The RNG belongs to the very broad family of *proximity graphs* [66]. Within this large family the RNG belongs to a hierarchy of distinguished planar proximity graphs established in [70], where it was shown that the RNG contains the minimum spannig tree (MST) as a subgraph, and is a subgraph of the Gabriel graph, which in turn is contained in the Delaunay triangulation (DT) (See also [53]). The MST, which contains the nearest neighbour graph [57], is a tree that spans all the points and has minimum total length [53]. The Gabriel graph (GG) may be defined analogously to the RNG provided that the lens determined by two points v_i and v_j is substituted for the smallest disc that contains these two points [53]. The Delaunay triangulation is a triangulation of the points with the property that every disc determined by three points that form the vertices of a triangle contains no other points. Thus the hierarchy of connected graphs is as follows:

MST ⊆ RNG ⊆ GG ⊆ DT

An illustrative example of this proximity graph hierarchy for a set of points in the plane is illustrated in Fig. 2. Note that, apart from the DT, the other three graphs capture some of the perceptual structure of the configuration of points.

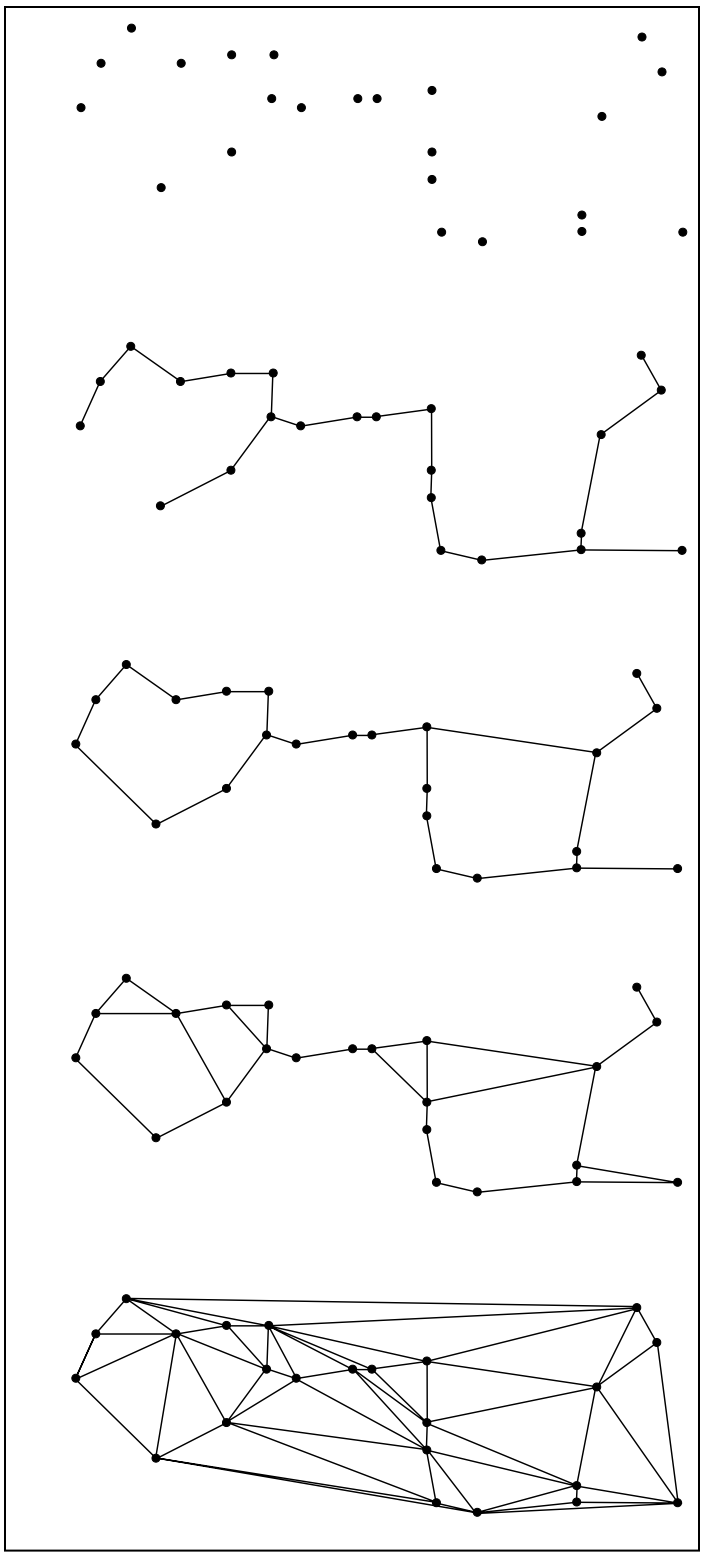

Figure 2. From top to bottom: A set of points, the Minimum Spaning Tree, Relative Neighborhood Graph, Gabriel Graph, and Delaunay Triangulation.

II. Applications

A. Vision

The MST in Fig. 2 (second from the top) has long been used as a powerful tool for detecting and describing Gestalt clusters, and is more powerful than the classical approaches to cluster analysis [87]. In the context of vision it is a simple bottom-up computational model of what David Marr called a *primal sketch* [52], [58], [67]. Nevertheless, this method suffers from a drawback when it is required to extract the perceptual structure of certain dot patterns. The MST "views" everything in the world as a tree, and hence perceived cycles in a pattern may not be realized. The RNG (third from top in Fig. 2) was proposed to remedy this wakness [70]. It "listens" to the data more than the MST: if the data are tree-like the RNG will mimic the MST, if the data points are distributed evenly on a circle it will output a cycle, and if the data are more densely scattered it will sometimes output the DT [70]. The GG (fourth from the top) contains a few more edges than the RNG, and is also a powerful model of a primal sketch. Finally, the DT (bottom) does not "listen" to the data, always contains the edges on the convex hull of the points, and hence fails (on its own) to extract the structure of dot patterns with no convex boundaries. Nevertheless, in conjuction with additonal techniques, the DT may also be very useful [9].

The MST, RNG, GG, and DT are all connected graphs, and therefore do not by themselves necessarily yield structures that accord with human perception, for data consisting of perceived disconnected components. One modification to handle such situations consists of deleting edges that are much longer than neighbouring edges [87]. However, this entails the tuning of parameters. An alternative approach consists of defining structures that do not have parameters that need to be tuned, such as the *sphere-of-influence* graph [67]. However, such graphs are beyond the scope of this paper.

B. Cluster Analysis

The seminal paper of Charles Zahn [87] that established the MST as as a powerful data mining tool, stimulated much research on the application of other graph-theoretic methods, such as MOSAIC [23], to the analysis of complex data that challenged traditional clustering methods [79]. Urquhart [77] applied the heuristics that Zahn employed with the MST, to the RNG, and concluded that better results are obtained when the MST is substituted with the RNG. Correa and Lindstrom [25] explore the family of empty-region proximity graphs known as β–skeletons, as tools for cluster analysis, and show that the RNG performs better than conventional approaches based on k-nearest neighbors or global scale parameters.

A recurring problem in cluster analysis is to determine for a given data set what the best number of clusters should be. In traditional methods the number of clusters is either specified in advance by the user, varies depending on the heuristic used, or yields a hierarchy (dendrogram) of all possible partitions.

Muhlenbach and Lallich [55] extend the work of Zahn [87] and Urquhart [77] by proposing a method, that incorporates the RNG, for finding the "ideal" number of clusters.

C. Shape Boundary Detection

A problem somewhat related to cluster analysis of point data is the estimation of the "boundary" of a dot pattern. Whereas the minimum spanning tree serves well as a model of the skeleton or "medial axis" of a line-like dot pattern, the concept of "boundary" refers to the external structure of a dense dot pattern, also called the *shape hull* [69] and *nonconvex hull* in the context of the computation of *scagnostics* in statistics [85]. If a dot pattern has "holes" it also has internal boundaries. The RNG has also been applied to the extraction of hole boundaries in dot patterns [75].

D. Visualization and Computer Graphics

When graphs contain a very large number of edges they may become impossible to visualize due to the fact that the limitations on line thickness tend to eliminate the blank area of the drawing, rendering an amorphous connected blob. In such cases a sparse subgraph of the original graph that preserves its fundamental shape may help to visualize the structure of the original graph. Gansner, Koren, and North [30] propose the RNG as a tool for this purpose. See also [79].

E. Archaeology

The study of spatial relationships in archaeology has been tackled recently using tools from graph theory (network analysis) [17]. Jiménez [38], and Jiménez and Chapman [39] applied the RNG to the exploration of meanigful relationships in point sets that represent archaeological sites or artefacts. They conclude that the RNG can reveal these contextual relationships more effectively than the standard methods based on linear distances, such as nearest neighbours.

F. Natural Computing

The term natural computing is used to mean several types of computation. Here it is used to mean that the computation is performed with natural materials or forces of nature (physical, chemical, or biological), such as gravity, DNA molecules, swarms, or bacteria. Andrew Adamatzky [6] has shown that when slime mold such as *plasmodium* is placed on a material surface that contains point locations with sources of nutrients, the foraging behavior of the mold generates the RNG in the form of a protoplasmic network. Furthermore, the experiments in [6] demonstrate that the mold constructs all the proximity graphs of the hierarchy in Equation 1, and in precisely the shown graph inclusion order. Slime mold has also been shown to compute the shape hull of a set of planar points [3].

G. Cosmology

One of the central problems in cosmology is estimating the cosmological parameters of the universe. A relevant property in this regard is the size and location of voids in the large-scale distribution of the galaxies. Ueda and Takeuchi [75] used the RNG in for the identification of two-dimensional voids. Furthermore, their algorithm was able to make semantic connections between this graph-theoretical approach and relevant physical interpretations.

H. Urban Planning

Daisuke Watanabe [83] applied proximity graphs as modesl of network construction to the study of grid road network patterns of the major cities in the United States, and found that the edges of the RNG include most of the grid roads. Furthermore, another study by Watanabe to evaluate both the configuration and travel efficiency of proximity graphs as transportation networks led to the conclusion that the network distance on the RNG is similar to the rectilinear distance in terms of both edge length and travel distance [84]. Aldous and Shun [8] introduce and motivate statistics computed on the RNG and other proximity graphs applied to route-length statistics with application to transportation networks. The RNG has also been computed using slime mould in the context of transportation networks such as highways [4], [5].

I. Machine Learning

In the area of machine learning, RNGs have been applied to several different aspects of the problem. One of the earliest applications is the reduction of the size of the training set [66]. For this purpose each and every point in the training data is first marked if it, and all its adjacent graph neighbours in the RNG, belong to the same class [50], [59]. The reasoning behind this approach is that such points are not close to the decision boundaries between the classes, and may thus be discarded without affecting these boundaries. It has been shown experimentally that the RNG can reduce the size of the training set without significantly decreasing classification accuracy [60].

The RNG has been applied to the design of nonparametric classification rules in instance-based learning [51]. In this context the classical *k*-nearest neighbour rule assigns an unknown instance by means of a majority vote amongst its *k* nearest neighbours, where *k* remains fixed after it is pre-tuned to the data at hand. The RNG-decision rule, on the other hand, assigns an unknown instance by means of a majority vote amongst its adjacent neighbours in the RNG (its relative neighbours). In this rule the number of neighbours used in each decision is not fixed in advance, and the choice of neighbours does not depend strictly on the distances themselves, but rather varies depending on the local density and geometric structure of the data around the instance to be classified [59].

One of the most powerful machine learning algorithms in terms of the classification accuracy is the *support vector machine* (SVM) [65]. Its drawback is that the complexity of training the SVM grows prohibitively as a function of the size of the training data. Hence it is desired to speed up the SVM with this particular goal in mind [29], [50].

Zighed, Lallich, and Muhlenbach [89] use the RNG to develop a statistical test they call the *separability index* for application to supervised learning. Their test is based on comparing the

relative weight of the edges in the RNG that connect data points of different classes, to the expected interval of a random distribution of the data labels on all the RNG edges. If the two values are not significantly different then no neighbourhood-based method will yield a reliable prediction model.

J. Adaptive Grid Generation

Grid generation is an essential technique used in numerical methods for solving partial differential equations. Berger [14] utilizes proximity graphs such as the minimum spanning tree to design data structures and algorithms for the automatic generation of adaptive subgrids in this context, and suggests that applying RNG's would yield better results.

K. Measuring Dissimilarity

In psychology, biology, anthropology, and the study of the evolution of cultural objects, such as musical rhythms, one is faced with the task of determining whether two dissimilarity matrices are significantly correlated. The classical approach to such an analysis calculates the statistical tests by taking into account *all* the pairwise dissimilarities, disregarding whether these dissimilarities are large or small. Levkovitch [47] proposed a modification of the classical methods by first computing the RNG of all the pairwise dissimilarities, and then calculating the statistical tests with only the most relevant dissimilarities, *i.e.*, those corresponding to edges in the RNG. He showed that many of the deficiencies of the classical approaches are largely overcome by icorporating the RNG.

In data-base indexing [31], image retrieval [32], and cluster analyses in general [10], the choice of the similarity measure employed may affect the results obtained, motivating the determination of which similarity measures yield similar results. Abdesselam and Zighed [1] propose a statistical test for this purpose based on a topology that incorporates the RNG. See also [88].

Adamatzky [2] studied how β–skeletons lose their edges as a function of β, and used this dependense to design a test to discriminare between random and non-random planar sets.

L. Percolation Theory

Percolation theory is useful for the description of a variety of physical phenomena, and it is also the simplest model of a physical process that displays a phase transition in porous materials. Consider the graph of an infinite square planar lattice consiting of vertices connected with edges (see Fig. 3), and color each vertex (*site* percolation) or edge (*bond* percolation) black, with probability p. The black vertices (edges) may be considered as "occupied" sites (edges). A central question in percolation theory concerns the properties of the connected components of the black vertices (edges) as a function of p. In particular, what values of p ensure that with probability 1 there exists a connected component of infinite size? The smallest such probabilities are referred to as site and bond *percolation thresholds*. There has been much interest in recent years in studying these thresholds for graphs other than simple lattices. For the RNG, Billiot, Corset, and Fontenas [16] established by analytic means the existence of nontrivial site and bond percolation thresholds, and Meichert [54] determined the assymptotic degree and diameter of RNGs, and obtained estimates of the site and bond percolation thresholds. Percolation properties have been investigated on other graphs related to the RNG, such as the Gabriel graph in [15]. See also [26] for further results along these lines.

M. Wireless Ad-Hoc Networks

A network is a graph consisting of nodes connected with edges, such as the minimum spanning tree. The nodes may stand for a variety of entities, such as computers, and the edges are traditionally physical electrical wires. A wireless network is a network without wires, such as a cellphone, Wi-Fi, or microwave network. In wireless ad-hoc (decentralized) networks all the nodes cooperate on an equal basis. In a mobile ad-hoc network (MANET) the nodes (devices) are mobile [13]. One of the central questions in such wireless communication networks is common to percolation theory: under what conditions on the location of the devices, their distance from each other (area of coverage), and the power of the nodes, does the network remain connected? Seddigh, Gonzalez, and Stojmenović [61] applied the RNG to the problem of minimizing the number of messages needed for broadcasting in wireless networks. Cartigny, Simplot, and Stojmenović [21] applied the RNG to minimum-energy broadcasting in ad-hoc networks with nodes of restricted battery power. See also [19], [20], [81], and [82] for alternate minimum-energy broadcasting protocols that make use of the RNG. The RNG has also been used to design "hole" avoiding protocols in wireless sensor networks [74].

Another central problem in sensor networks is the calculation of the coverage, a parameter that directly affects the quality of the service provided to the user. Li, Wan, and Frieder [48] provide an efficient distributed algorithm for calculating the best coverage, in which each sensor node locally constructs all edges in the RNG that are incident to that node. If every node in a wireless ad-hoc network has the same transmission radius, and the transmission radius is equal to the longest edge in the RNG, then the RNG can be locally constructed. Thus, the longest edge is called the *critical transmission radius*. Yi, Wan, Wang, and Su [86] compute asymptotic bounds on the critical transmission radius for the case when the network is modeled as a Poisson point process.

N. Graph Theory

The original definition of the RNG for points in the plane with the Euclidean metric, and using the lens as a region of neighbourhood [70], stimulated research to explore the graph-theoretical properties of the RNG [76], and to generalize this graph using a variety of alternate definitions of neighbourhood regions, with non-Euclidean metrics, in higher dimensional spaces [36], [56].

Toussaint [71] and ElGindy and Toussaint [28] generalized the RNG to the setting in which the points are vertices of a simple polygon, and visibility is taken into account when testing for lens inclusion. They proposed this generalization as

a tool for decomposition of polygons into simpler components in the context of pattern recognition. Similar extentions of the RNG were proposed and investigated by T.-H. Su and R.-C. Chang [62]. These RNG's are referred to as *constrained relative neighbourhood graphs* (CRNG). Lingas and Mukhopadhyay considered the CRNG of a special type of simple polygon known as a *histogram*, and showed that it can be computed in O(n) worst-case time.[1] Cimikowski obtains several combinatorial properties of a variant of the RNG, which he dubs RCG [24]. Specifically, a graph G(V, E) is an RCG iff there exists an edge e_{ij} of length d_{ij} between v_i and v_j iff $d_{ij} < \max(d_{ik}, d_{jk})$, $\forall\ k \neq i, j$, $i \neq j$.

Kirkpatrik and Radke [45], Urquhart [77],Veltkamp [80], and Cardinal *et al.*, [18], generalized the region of influence between two points by parameterizing the lens in Fig. 1 so that it can be either larger or smaller that the lens.

Ichino and Sklanski [34] proposed that the region of influence between two points be the smallest isothetic hyperbox that contains the two points. This region has all its edges parallel to the axes of the space, and enjoys the property that the neighborhood relations are invariant to scaling of the axes coordinates.

Urquhart [78] obtained a variant of the RNG (the Urquhart graph) by discarding an edge e from the DT if neither of the other two vertices that make up the two triangles in the DT that share e, are conatined in the lens of the endpoints of e. Since other vertices of the DT may still be included in the lens of e, the resulting graph (UG) is a supergraph of the RNG. See also [11], and [68] for comments on the UG, and how well it approximates the RNG.

Chang, Tan, and Lee [22] generalized the definition of relative neighbors used in the RNG to obtain the k-RNG. Recall from Fig. 1 that in the RNG two points v_i and v_j are relative neighbors provided that the lens determined by v_i and v_j does not contain any other point of the set. In the k-RNG two points v_i and v_j are k-relative neighbors provided that the lens determined by v_i and v_j contains less than k points of the set. Clearly, with this notation, the RNG is the 1-RNG, and when $k > n$-3 the k-RNG is the complete graph. The authors obtain several properties of k-RNGs. Since complete graphs, and hence, k-RNGs with $k > n$-3 are Hamiltonian (admit a Hamiltonian cycle), and there exist RNGs that have no cycles, it is of interest to determine what is the least value of k for which the k-RNG is Hamiltonian. It is established in [22] that 20-RNGs are Hamiltonian.

Another generalization along these lines is the *witness* RNG that depends on two classes of points: the vertices V of the resulting graph, and the witnesses W that impose constraints on the presence of the edges [27]. The witness RNG of a point set V, with respect to a set W of witnesses, is the graph with vertex set V in which two points v_i and v_j in V are adjacent if and only if their lens does does not (in the negative version) contain a witness point. In the negative RNG witness graph (WRNG), witnesses remove edges between the pairs of points. This witness concept has also been generalized to other proximity graphs [12].

Since the planar RNG is a subgraph of the planar DT, it contains a linear number of edges. In three dimensions on the other hand the DT may have a quadratic number of edges. However, Agarwal and Matousek [7] proved that in three dimensions the RNG has O($n^{4/3}$) edges.

III. Computational Complexity

The naïve brute-force algorithm for computing the RNG of n points constructs the lens for each of the $\Theta(n^2)$ pairs of points, and then tests the remaining n-2 points for inclusion in each lens, thus leading to an O(n^3) worst-case computational complexity. An O(n^2) time algorithm was obtained in [70] by using the properties that the RNG is a subgraph of the DT, and that the DT contains a linear number of edges, and can thus be computed in O(n log n) worst-case time. This means that the only lenses that need to be tested for emptiness are those corresponding to the edges of the DT. Toussaint and Menard [72] showed using bucketing techniques that this O(n^2) time algorithm can be made to run in approximately O(n) expected time for input points uniformly distributed in the unit square.

J. Katajainen and O. Nevalainen [43] obtained an O(n^2) time algorithm by first computing a supergraph of the RNG (the *geographic neighbourhood graph*), which contains O(n) edges, and then deleting the extra edges. They also show that if bucketing is applied the algorithm runs in O(n) expected time for points generated from a homogenous planar Poisson process. See also [41] for a similar approach.

The fact that the DT can be computed in O(n log n) worst-case time suggests that the RNG can be computed in o(n^2) time. Suppowit [64] obtained an O(n log n) time algorithm for the RNG by first computing the DT in O(n log n) time, and then carefully eliminating edges from the DT. He also proved an $\Omega(n \log n)$ time lower bound on the complexity of computing the RNG, thereby establishing the optimality of his algorithm. This approach was improved by N.-F. Huang [33] who showed that the RNG could be computed directly, via divide-and-conquer, in O(n log n) time while the DT is being constructed. Lingas [49] showed that the RNG could be computed directly from the DT in linear time.

Another kind of triangulation is the 30º triangulation, which has the property that all its angles are greater than or equal to 30º. Keil, Tzvetalin, and Vassilev [44] show that the RNG is a subgraph of every 30º triangulation.

For the case of three dimensions, Agarwal and Matousek [7] present randomized algorithms for computing the RNG that run in O($n^{7/4+\varepsilon}$) expected time for arbitrary points, and in O($n^{3/2+\varepsilon}$) expected time for points in general position. They also point out that these algorithms can be made deterministic with the same time complexities. See also [35].

Joe O'Rourke [56] explored the computational aspects of the RNG for the L_1 and L_∞ metrics. In particular, he showed that the RNG with the L_1 metric in two dimensions, and with the L_∞ metric in two and higher dimensions may be computed in O(n^2 log n) worst-case time. D. T. Lee [46] improved the

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complexity for the L_1 metric in two dimensions to O(n log n). Jaromczyk and Kowaluk [36] give an O(n^2) alsgorithm that computes the RNG in d dimensional spaces with L_p metrics for $1 < p < \infty$ when no three points of the input set form an isoceles triangle. See also [40] and [41] for additional algorithms that compute the RNG with L_p metrics.

Su and Chang [63] propose an algorithm for computing the k-RNG in time O($n^{5/3}$ log n) for fixed values of k.

IV. Concluding Remarks

That the RNG is so successful at extracting meaningful information from data in so many widely disparate domains of knowledge is remarkable, and suggests that the RNG may be a good candidate for a universal model of neighborhood. This provides motivation for developing software implementations of practically efficient and robust algorithms for computing RNGs in all dimensions for a wide variety of measures of distance and dissimilarity. It is noteworthy that the only regular tilings that exist in the plane, the triangular, square, and hexagonal tessellations in Fig. 3, are all realized by the RNGs of their lattice points.

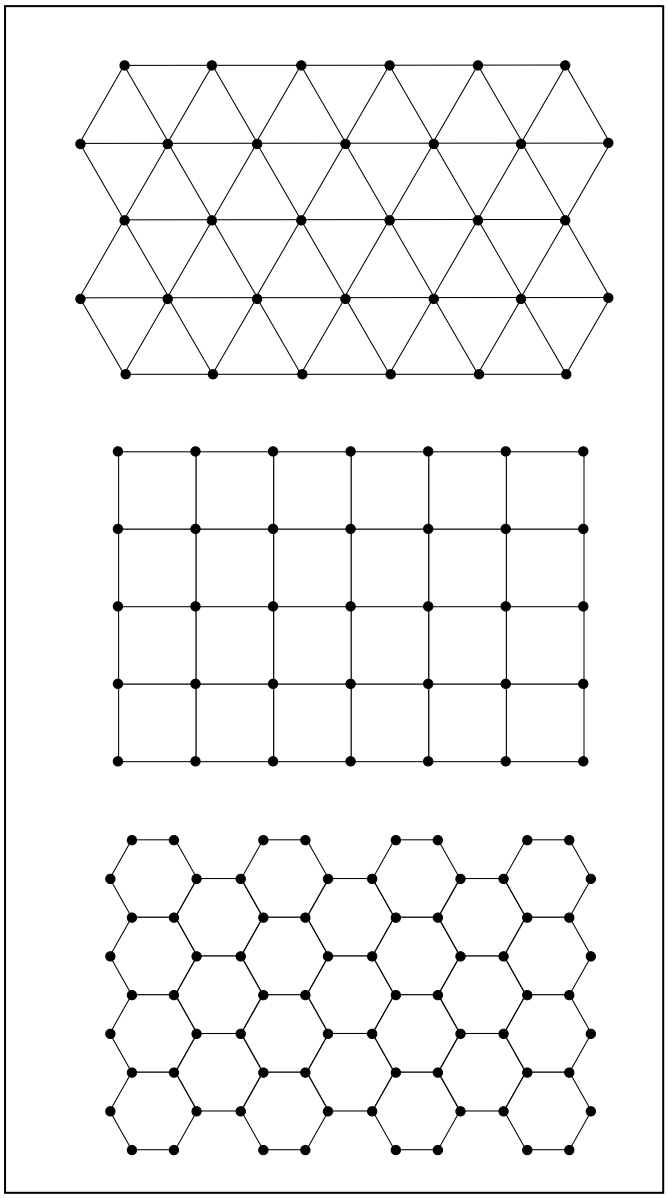

Figure 3. The three regular lattices are the RNGs of their lattice points.

On the theoretical side, there remain some open problems concerning tight upper and lower bounds on the number of edges in the RNG of n points. In the two dimensional Euclidean space (d=2) the well-known upper and lower bounds that follow from the theory of planar graphs are, respectively, 3n-6 and n-1. For d $\geq$ 4 the number of edges in the RNG is $\Omega(n^2)$ [37]. The intriguing case is for three dimensions, where obtaining sharp bounds remains a challenging open problem.

Recall that Chang, Tan, and Lee established in [22] that 20-RNGs admit a Hamiltonian cycle. It remains an open problem to determine whether the number 20 may be reduced. It is also shown in [22] that the k-RNG has O(kn) edges, which leads to an algorithm for computing the k-RNG in O(kn^2) time. It is not known if this algorithm is optimal.

Berger [14] applied the minimum spanning tree to design data structures and algorithms for the automatic generation of adaptive subgrids in the numerical computation of partial differential equations. It remains to be determined if applying RNG's instead, would yield better results.

The measures of distance used in most of the applications reviewed here consist predominantly of Minknowski metrics (L_p norms). However, the fundamental idea behind the RNG may be generalized to other metrics, non-metrics, and more general dissimilarity measures, such as the number of links between nodes in a graph, or the degree of separation between people in a variety of types of social networks. The RNG has not yet been explored in this domain. The success enjoyed by the RNG in other areas, suggests that extracting the RNG from social networks may provide useful information concerning phenomena such as the propagation of ideas, products, or diseases.

A general pattern concerning open problems and future research that emerges in most of the applications outlined here is that the RNG serves well as a preliminary step that has some limitations ascribed to the fact it is based exclusively on spatial information. In most applications, what is needed to make the RNG even more powerful, in addition to this purely geometric information, is the incorporation of complementary application-specific higher level knowledge. On this front there is still plenty of work to be done.

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