Round off Error propagation in simulation of RC circuit

Subtitle: Simulation of RC circuit

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Abstract— **Electrical devices often use RC, LC or RLC circuit to design power amplifiers, filters and mixers etc. Complex iterative algorithms are used for the simulation of these circuits. . This paper illustrates numerical experiments on growth of rounding off error in simulation of simple RC circuit using fourth order Runge-Kutta method (RK) and the Runge-Kutta-Fehlberg with adaptive step size control method (RKF). Our analysis indicates that in simulations with RK method, round off error grows ~ 80% with 10-15 iterations and ~96-98% within 100 iterations with different step sizes and double precision. In the simulations with RKF method round off error grows to 70% with 10-20 iterations and ~80% within 100 iterations. It does not exceed 80% for 1000 of iterations for single and double precision. This indicates that growth of round off error in RKF method is less and it should be used to minimize round off error.**

Keywords—Simulation of RC circuit, Round off error, Runge Kutta methods

I.   Introduction

In the era of powerful computers, the errors of numerical integration are the main limitation in the modeling of dynamical systems (Milani and Nobili 1988). For the long-term behavior, it is known that integration errors grow with respect to time/number of iterations, using traditional integrators such as the Runge-Kutta methods (Hairer et al., 1993). Number of papers has reported that the growth rate becomes linear if simplestic integrators are used (Sanz-Serna 1992; Hairer and Stoffer, 1997; Hairer and Gustaf, 2005). The integration errors are composed of the truncation and the round-off errors. After the truncation error has been made insignificant by means of symmetric linear multistep methods, the round-off error will emerge and start to play a key role (Petit 1998). Previous studies indicate that to minimize the round off error one needs to avoid adding huge number to small number, avoid subtracting numbers that are close, minimize the number of arithmetic operations involved (Widrow and Kollár, 2008 and references therein). The growth of round off error while using numerical integrations has been reported in the variety fields e.g simulation of weather forecasting, agriculture, electronic circuits, signal processing, stock market, billing of mobile phones, astrology etc. (Nastov et al., 2007; Janakiraman et al., 2000; Khalkho et al., 2013; Khalid et al., 2007; Goel, and Dasha, 2007; Widrow and Kollár, 2008).

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With the advancement of technology there is increasing demand of high performance portable communications system. This imposes the need for simulation algorithms those can be used efficiently and accurately (Nastov et al., 2007). Runge kutta methods are widely used in simulation algorithms. While writing algorithms using Runge Kutta method round off error may propagate and affect output (Khalid et al., 2007). Recently few papers have suggested algorithms for simulations of electric/electonic circuits (Nastov et al., 2007; Oliveira et al 2006). These papers suggest that iterative computations may affect output.

The RC, LC or RLC circuits are commonly used in electronic devices. The RC circuit is used to protect against power outages. RC-coupled amplifiers are used very often in circuits with vacuum tubes or discrete transistors. The RF communication circuits such as mixers, switched-capacitor filters, and amplifiers also contain combination of RC/RLC circuits. Hence in this paper we present examples of round off error growth in simulation of simple RC circuit with different step sizes for single and double precision. This paper is organized as: section 2 describes fourth order Runge Kutta method (RK) and the Runge-Kutta-Fehlberg method with adaptive step size control (RKF). Concept of RC circuit is explained in section 3. Section 4 illustrates propagation of error due to rounding off in simulation of RC circuit using RK and RKF method. Conclusions are made in section 5.

II. The Runge-Kutta methods

The Runge-Kutta methods are an important family of iterative methods for the approximation of the solutions of ordinary differential equations. These methods were developed around 1900 by the German mathematicians Runge and Kutta. Details of Runge Kutta methods are reported by Bogacki and Shampine (1989), Gear, (1971), Shampine, and Watts (1977).

A. Fourth order Runge-Kutta method

The classical Runge Kutta (RK) method is given by Dormand and Prince (1980) and Griffiths and Higham (2010). Consider the problem Y'(t)=f(t,y)

$Y(t_0)=\alpha$

Define h to be the time step size and $t_i = t_0 + ih$. Then the following formula

$\omega_0 = \alpha$

$k_1 = hf(t_i; \omega_i)$

$K_2 = hf(t_i+h/2; \omega_i+k_1/2)$

$K_3 = hf(t_i + h/2; \omega_i + k_2/2)$

$K_4 = hf(t_i + h/2; \omega_i + k_3)$

$\omega_{i+1} = \omega_i + 1/6(K_1 + 2k_2 + 2K_3 + k_4)$

Computes an approximate solution, that is $\omega_i \approx y(t_i)$.

B. the Runge-Kutta-Fehlberg method with adaptive step size control

The Runge-Kutta-Fehlberg (RKF) method is an example of adaptive time-stepping method. It uses a fourth-order and fifth-order Runge-Kutta method that share some evaluations of f(t; y), in order to reduce the number of evaluations of f per time step to six, rather than the ten that would normally be required from a pairing of fourth and fifth-order methods. In this method we will use adaptive step size control during the computation. The idea is to start with a moderate step size. When we detect that the expected error is larger than ε, then we reduce the step size and recalculate the current step. When we detect that the expected error is less than ε, then we keep the current step and slightly enlarge the step size in the next step. This requires us to have a good estimation of the expected error (Fehlberg, 1969). The flow chart describing adaptive step size control is given in Fig. 1.
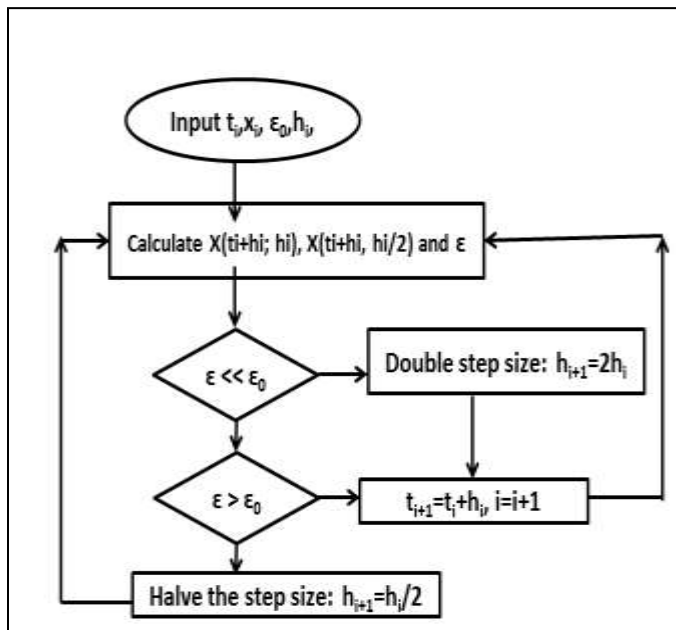

Fig. 1 Flow diagram of the step size control by use of the step doubling method

The RKF method is explained below.

$\omega_0 = \alpha$

$k_1 = hf(t_i; \omega_i)$

$K_2 = hf(t_i + h/4; \omega_i + k_1/4)$

$K_3 = hf(t_i + 3h/8; \omega_i + 3/32K1 + 9/32K2)$

$K_4 = hf(t_i + 12h/13; \omega_i + 1932/2197*K_1 - 7200/2197*K_2 + 7296/2197*K_3)$

$K_5 = hf(t_i + 12h/13; \omega_i + 439/216*K_1 - 8*K_2 + 3680/513*K_3 - 845/4104*K_4)$

$K_6 = hf(t_i + 12h/13; \omega_i - 8/27*K_1 + 2*K_2 - 3544/2565*K_3 + 1859/4104K_4 - 11/40*K_5)$

$\omega_{i+1} = \omega_i + 25/216*K_1 + 1408/2565K_3 + 2197/4104K_4 - 1/5*K_5$

$\acute{\omega}_{i+1} = \omega_i + 16/135*K_1 + 6656/12825*K_3 + 28561/56430*K_4 - 9/50*K_5 + 2/55*K_6$

$R = 1/h|\acute{\omega}_{i+1} - \omega_{i+1}|$

$\delta = 0.84 \left(\dfrac{\varepsilon}{R} \right)^{1/4}$

if $R \leq \varepsilon$ keep ω as the current step solution and move to the next step with step size δh

if $R > \varepsilon$ recalculate the current step with step size δh.

III. R-C Circuit

A resistor–capacitor charging circuit (RC circuit), or RC filter or RC network, is an electric circuit composed of resistors and capacitors driven by a voltage or current source as shown in Fig. 2. A first order RC circuit is composed of one resistor and one capacitor with voltage source arraged in series. RC circuits can be used to filter a signal by blocking certain frequencies and passing the others. The most common RC filters are the high-pass filters and low-pass filters. Bband-pass filters and band-stop filters usually require RLC filters, though crude ones can be made with RC filters
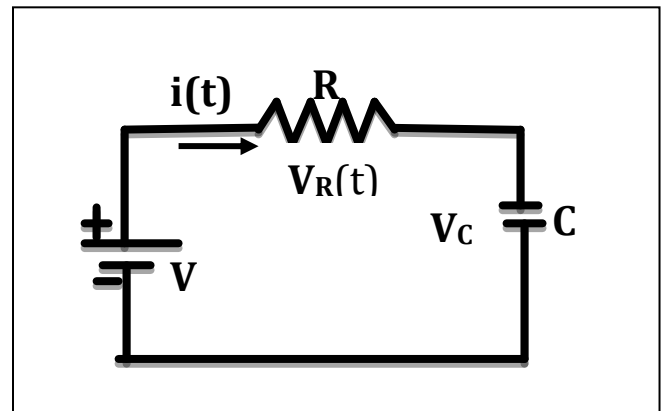

Fig.2   A typical resistor–capacitor circuit (RC circuit). V is source voltage. I(t) represents currents at time t, $V_R(t)$ stands for voltage across resistor 'R' at time t. V_C represents voltage across capacitor 'C'

The typical RC circuit is shown in Fig.2 If the charge on the capacitor is Q and the current owing in the circuit is I, the voltage across R and C are RI and Q/C respectively.

Kirchoff's law says that the voltage between any two points has to be independent of the path used to travel between the two points,

$$RI(t) + (1/C). Q(t) = V(t)$$

Assuming that R; C and V are known, this is still one differential equation in two unknowns, I and Q. However the two unknowns are related by $I(t) = dQ/dt(t)$

so that RQ'(t) +(1/C). Q(t) = V (t)

The V = 0 Solution

If the applied voltage V = 0, this equation is separable and consequently easily solved.

$$R\frac{dq}{dt} = -\frac{1}{c}Q$$

$$\frac{dq}{Q} = -\frac{1}{RC}dt$$

$$\int \frac{dq}{Q} = -\frac{1}{RC}\int dt$$

Applying log on both the sides

$$\ln|Q| = \frac{1}{RC}t + K$$

$$Q(t) = K.e^{-t/RC}$$

where $K = \pm e^k$. At t = 0, Q(0) = Ke^0= K, so $Q(t) = Q(0)e^{-t/RC}$. Hence the capacitor just discharges exponentially through the resistor. When V is nonzero, the equation is no longer separable. But there is another approach that allows us to solve $RQ'(t) + \frac{1}{c}Q(t) = V(t)$.

Divide by R we get

$$RQ'(t) + \frac{1}{RC}Q(t) = \frac{1}{R}V(t) \qquad (1)$$

Multiply whole equation by µ(t).

µ(t)*Q'(t)+1/RC*µ(t)*Q(t)=1/R*µ(t)*V(t)   (2)

Apply derivative on left hand side and substitute µ'(t)=1/RC*µ(t). This equation for µ(t) is separable and hence may be solved by the same technique that we used to solve

$$Q'(t) = \frac{-1}{RC}Q(t).$$

Now substitute $\mu(t) = e^{t/RC}$.

Multiply equation (1) by $e^{t/RC}$ gives

$$e^{t/RC}Q'(t) + \frac{1}{RC}e^{t/RC}Q(t) = \frac{1}{R}e^{t/RC}V(t)$$

$$\frac{d}{dt}\left(e^{t/RC}(t)\right) = \frac{1}{R}e^{t/RC}V(t)$$

$$e^{t/RC}Q(t) = \frac{1}{R}\int e^{t/RC}V(t)dt$$

$$Q(t) = \frac{1}{R}e^{-\frac{t}{RC}}\int e^{\frac{t}{RC}}V(t)dt$$

If V(t)=V_0 Then

$$\int e^{t/RC}V(t)dt = \int e^{t/RC}V_0 dt = RCV_0 e^{t/RC} + k,$$

therefore

$$Q(t) = CV_0 + Ke^{-t/RC} \qquad (3)$$

Where K=k/R is an arbitrary constant (Meijs, 2008).

IV.   Results and Discussions

We simulated RC circuit described in section 3. Simulations have been carried out with two methodologies (1)

RK method and (2) RKF method as described in section 2. These experiments are conducted for different step sizes for single and double precisions. We track round off error for 1000 time steps/iterations and for step size h=0.001, h=0.01, h=0.05 with same initial conditions for both the methods. Figure 3 illustrates increase in round off error (expressed in percentage) versus number of time steps/iterations for step size h=0.001.
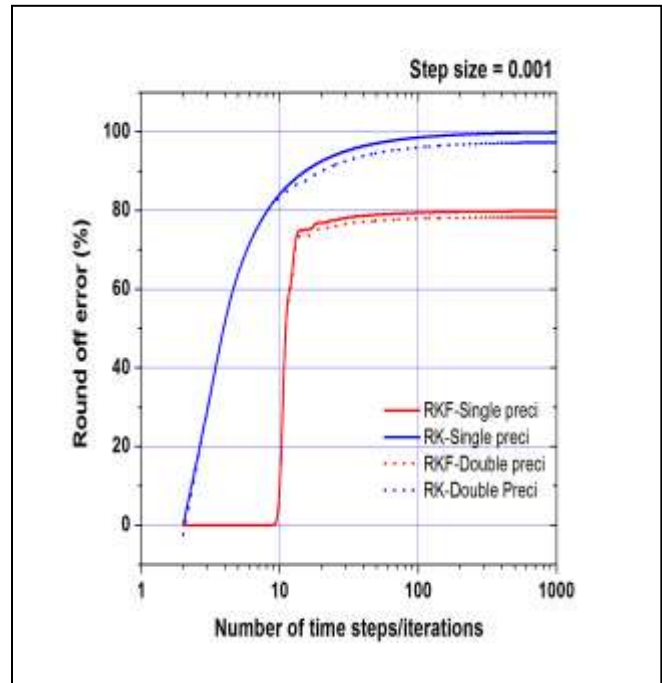

Fig.3.   Growth of round of error versus number of steps/iterations in simulation of RC circuit for step size= 0.001, for RK single precision (solid red line), RK double precision (dotted red line),   RKF single precision (solid blue line), RKF double precision (blue dotted line).

In case of RK method, round off error grows rapidly to ~85 % within 10 time steps/iterations and ~98 % within 100 time steps/iterations. For RKF method round off error is negligible for the initial 10 time steps/iterations. It then increases to ~77% within next 10 time steps/iterations. Due to adoptive size control round off error does not exceed 80% for 1000 time steps/iterations. These experiments are repeated for double precision indicated by dotted lines. It shows that round off error is less in double precision than in single precision (but shows same nature as for single precision).

Figure 4 shows percentage increase of round off error with number of time steps/iterations for both the methods and for the step size h=0.01. It shows that for RK method round off error increases to ~83% within first 10 time steps/iterations and ~98% within 100 time steps/iterations. In case of RKF method round off error is negligible for initial 6 time steps/iterations it then increase to 68% within 10 times steps/iterations. It increases to 80% within 100 time steps/iterations and does not exceed later. The round off error

is less for double precision (indicated by dotted lines) than the single precision
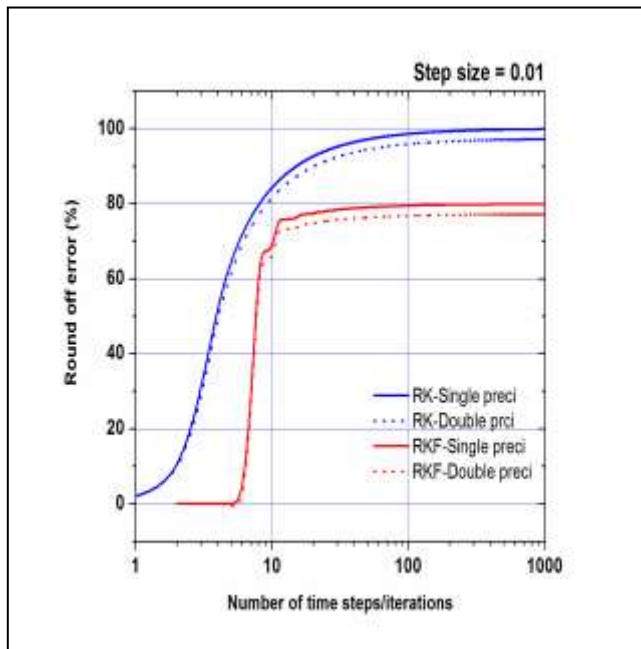

Fig.4. Growth of round of error versus number of time steps/iterations in simulation of RC circuit for step size= 0.01; for RK single precision (solid red line); RK single precision (dotted red line); RKF single precision (solid blue line) and RKF double precision (blue dotted line).
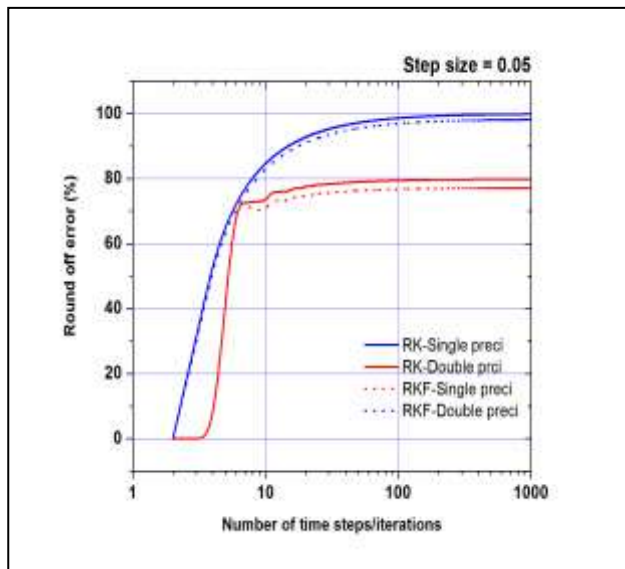

Fig.5 Growth of round of error versus number of time steps/iterations in simulation of RC circuit for step size= 0.05; for RK single precision (solid red line); RK double precision (dotted red line), RKF single precision (solid blue line), RKF double precision (blue dotted line).

The percentage increase of round off error with number of time steps/iterations for both the methods and for the step size h=0.05 is shown in Figure 5. In RK method round off error grows to ~82% within first 10 time steps/iterations it then increase to ~96% within 100 time steps/iterations and ~100% within 1000 time steps/iterations. For RKF method round off error growth is higher ~75% within first 10 time steps/iterations for the step size h=0.05 as compared to other smaller step sizes. It then grows to 80% within 100 time steps/iterations.

The Figures 3-5 show that round off error is higher for smaller step sizes for RK method, while it smaller in case of RKF method for first 100 time steps/iterations. It increases to ~98-99% within 1000 time steps/iterations for simulation with RK method and ~80 with RKF method.

V. Conclusions

We simulate Resistance-Capacitor (RC) circuit with fourth order Runge-Kutta method (RK) and the Runge-Kutta-Fehlberg method with adaptive step size control (RKF). Numerical experiments are conducted for different step sizes and for single/double precisions. Our analysis indicate that in simulations with RK method, round off error grows ~ 80% with 10-15 time steps/iterations and ~96-98% within 100 time steps/iterations with different step sizes; for single and double precisions. For both the methods, round off error is less in simulations with double precision as compared to single precision for different step sizes. For RK method the round off error is higher for smaller step sizes, while it smaller in case of RKF for first 100 time steps/iterations. Simulations with RKF method round off error grows to 70% with 10-20 time steps/iterations and ~80% within 100 time steps/iterations. It does not exceed 80% for 1000 of iterations for single and double precision. This indicates that growth of round off error in RKF method is less. The above results are pertaining to fourth order Runge-Kutta and Runge-Kutta-Fehlgerg method with adaptive step size control. There may be a few more methods used by other researchers and a few more may evolve in future. We hope that researchers will be able to find comprehensive treatment to keep the round off error minimal.

Acknowledgment

References

A. Milani; M. Anna Nobili, "Integration error over very long time spans", Ramsau, Austria, Mar. 14-18, vol. 43, no. 1-4, 1988, p. 1-3.

B. Widrow, and Kollár, I., Quantization Noise: Roundoff Error in Digital Computation, Signal Processing, Control, and Communications, Cambridge university press., pp 1-742, New York, 2008.

Bogacki and L. F. Shampine, "A 3(2) pair of Runge-Kutta formulas," Applied Mathematics Letters, vol. 2, no. 4, 1989, pp. 321–325.

C.W. Gear, "Numerical Initial Value Problems in Ordinary Differential Equations" Englewood Cliffs, NJ: Prentice-Hall, Chapter 2, 1971.

D.,Khalkho, N. S. Raghuwanshi, S. Khalkho and R. Singh, "A computer model for designing of permanent gully control structures", African Journal of Agricaltural Research, vol. 8(29), 2013, pp 3860-3872.

David F. Griffiths, Desmond J. Higham, Numerical Methods for Ordinary Differential Equations, Springer Undergraduate Mathematics Series ISSN 1615-2085, ISBN 978-0-85729-147-9, Springer London Dordrecht Heidelberg New York, 2010, pp 1-267.

E. Fehlberg, "Low-order Classical Runge-Kutta Formulas with Stepsize Control and Their Application to Some Heat Transfer Problems", NASA Technical Report, NASA TR R-315, 1969.

E. Hairer and S derlind Gustaf, "*Explicit, time reversible, adaptive step size control*" , SIAM J. Sci. Comput. Vol . 26, 2005, p.1838-1851.

E. Hairer, S-P. Nørsett, G. Wanner, Solving ordinary differential equations I, Nonstiff problems 2nd ed., Berlin: Springer Verlag, ISBN 978-3-540-56670-0, 1993.

E. Hairery and Daniel Stoffer, "REVERSIBLE LONG-TERM INTEGRATION WITH VARIABLE STEPSIZES", SIAM J. SCI. COMPUT., Vol. 18, No. 1, pp. 257-269, 1997

J. M., Sanz-Serna "Symplectic integrators for Hamiltonian problems: an overview", Acta Numerica vol. 1,1992, p. 243-286.

J. Oliveira and A. Araújo, "Envelope transient simulation of nonlinear electronic circuits using multi-rate Runge-Kutta algorithms", WSEAS Transactions on Electronics, Volume 3, Issue 2, 2006, pp. 77-84.

J.R. Dormand and P.J. Prince, "A family of embedded Runge-Kutta formulae, Journal of Computational and Applied Mathematics, volume 6, no 1, 1980, p19-26 .

J. M. Petit, "Symplectic Integrators: Rotations and Round off Errors", Celestial Mechanics and Dynamical Astronomy, v. 70, Issue 1,1988, p. 1-21.

L.F. Shampine, and H. A. Watts, in Mathematical Software III, J.R. Rice, ed. (New York: Academic Press), 1979, pp. 257–275.

O. Nastov, R. Telichevesky, K. Kundert, and J. White, "Fundamentals of Fast Simulation Algorithms for RF Circuits", Proceedings of the IEEE, Vol. 95, No. 3, 2007, p 600-621.

P. van der Nick Meijs., "Model Order Reduction of Large RC Circuits", Model Order Reduction: Theory, Research Aspects and Applications Mathematics in Industry Volume 13, 2008, pp 421-446.

S. Goel, S.K. Dash, ¨Response of model simulated weather parameters to round-off-errors on different systems:, Environmental Modelling and Software, 2007, P. 1164–1174.

S. Janakiraman, J.V. Ratnam and A. Kaginalkar, "Study of machine round-off response on weather forecasting simulations using high performance computing systems, High Performance Computing in the Asia-Pacific Region", Proceedings, The Fourth International Conference/Exhibition vol. 2, 14-17 May, 2000.

S. Khalid, R. Amin, M. Hossain and M. Anwer, "Numerical round-off error in cellular phone services billing system", IEEE,1-4244-1551-9/07,2007.

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He has developed a penchant for research and roots lay in the inspiration at home. His mother, Dr. S. Fadnavis is an eminent scientist and aptitude for research is a family bloodline. He got a great mentoring at home from his mother, who is also an expert on numerical chaos and has a lot of international publications to her credit. Shreyas grew up in a research environment and thus could build the ability to research at such a younger age. Shreyas was always fascinated by computing techniques and hence opted for a specialized course in computer science even in his high school period.