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A Numerical Solution of Burgers' Equation Based on Multigrid Method

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***Abstract*—in this article we discussed the numerical solution of Burgers' equation using multigrid method. We used implicit method for time discritization and Crank-Nicolson scheme for space discritization for fully discrete scheme. For improvement we used Multigrid method in fully discrete solution. And also Multigrid method accelerates convergence of a basics iterative method by global correction. Numerical results confirm our theoretical results.**

***Keywords*—multigrid method, V cycle, Burgers' equation, numerical scheme, Taylor series.**

I. Introduction

The study of nonlinear problem is very crucial job in the field of nonlinear science and engineering. The 1-D non-linear partial differential equation is analogous to the one dimensional Navier-Stokes equation excluding the stress term and it was published in 1940 by J.M. Burgers. This 1-D model used for the solution of Navier-Stokes equation, and also applied for the solution of laminar and turbulence flows. An interesting test case with shock generation is provided by the time evolution of a sinusoidal wave and last five decade researcer are working to accelerate the convergence of steady state. For fully discritization, we used backward difference method and Crank-Nicolson method to discritize the time direction and space direction, respectively. To reduce the nonlinearity term we have applied Taylor series expansion. And also Crank Nicolson discritization scheme is used to solve the Burgers' equation, see in [6]. Multigrid method have been proved very successful in accelerating the convergence of elliptic system [1, 3 & 5]. Relaxation techniques is very efficient in removing the error component with wavelength comparable to mesh size [1]. Multigrid method exploit the different grid size by solving the problem on different discretizations. Computational cost of multigrid method is proportional to the size of problem and it also applicable for higher dimension. Multigrid method is a method of mixing relaxation sweep with numerical solution and residual of coarse grids [4].

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Traffic flow model have many application in defence such as traffic analysis for military intelligence counter intelligence and computer networking.

II. Preliminary

In this section, we discuss some preliminary to be used in the subsequent sub section.

Consider traffic flow model on the Indian road highway. There are two parameters for traffic flow model such as velocity of car, density of car and traffic flow. Average number cars passing per hour per lane is called the traffic flow. Number of cars per mile is called density of the cars.

Traffic flow = (Traffic density)* Velocity field. In mathematically it represents

$$f(u) = \rho u. \qquad (2.1)$$

Cars will be conserved on the highway, so by continuity equation it derived as follows:

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in R, t > 0, \qquad (2.2)$$

Where $f(u) = \frac{u^2}{2}$. (2.3)

Using (2.1) and (2.3) in (2.2), we arrive at

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0. \qquad (2.4)$$

The equation (2.4) is known as Burgers' equation. The equation (2.4) could serve as a nonlinear analog of the Navier-Stokes equations. The equation (2.4) have a convective term, a diffusive term and a time-dependent term. When density of the car is very large, we will add the viscous term in equation (2.4). After adding viscous term in (2.4), we obtain

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \mu u_{xx}, \qquad (2.5)$$

where u is the speed of traffic flow, μ is the kinematic viscosity, x is the spatial coordinate and t is the time. The equation (2.5) is parabolic when the viscous term is included. If the viscous term is neglected, the remaining equation is hyperbolic. If the viscous term is dropped from equation (2.5) the nonlinearity allows discontinuous solutions to develop.

III. Multigrid method

Classically multigrid method was developed in a fast and efficient way to solve algebraic system of equation resulting from discritization of partial differential boundary-value problems [4]. The multigrid technique consists of solving the algebraic system using a succession of grids of increasing mesh sizes. Subsequently multigrid method [1, 3 & 5] has also been extended to non-elliptic system. Basic multigrid method like the correction scheme and the full approximation scheme was developed by Brandt [2] for solving boundary-value problems.

Algorithm

Two-grid V-cycle [3] :

- Iterate on $A_h.u = b_h$ to u_h (say 3 Jacobi or Gauss-Seidel Step).
- Restrict the residual $u_h = b_h - A_h.u_h$ to the coarse grid by $r_{2h} = R_h^{2h}.r_h$.
- Solve $A_{2h}.E_{2h} = r_{2h}$ (or come close to E_{2h} by 3 iteration from E = 0.
- Interpolation E_{2h} back to $E\_h = I_{2h}^h.E_{2h}$. Add E_h to u_h.
- Iterate 3 more times on $A_h.u = b_h$ starting from the improved $u_h + E_h$.

  $A = A_h$ = original matrix.

  $R = R_h^{2h}$ = restriction matrix.

  $I = I^h_{2h}$ = interpolation matrix.
- Step 3 involves a fourth matrix A_{2h} to be defined now. A_{2h} is square and it is smaller than the original A_h.

  The coarse grid matrix is $A_{2h} = R_h^{2h}.A_h.I^h_{2h}$.

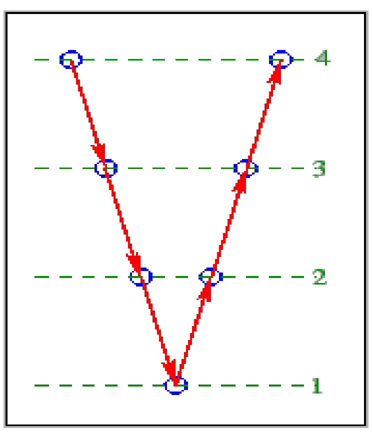

Figure 1. V-cycle

IV. Full Discretization of Equation

In this subsection, we have used the implicit scheme for time derivative and Crank-Nicolson scheme [7] for space to derivative to discritize the equation (2.5). Applying implicit scheme to nonlinear equation (2.5) is not quite as straightforward as for linear equation. Using backward difference scheme in time direction and central difference in space direction in equation (2.5), we obtain

$$\frac{\Delta u_i^{n+1}}{\Delta t} = -L_x \frac{(F_i^n + F_i^{n+1})}{2} + \mu L_{xx} \frac{u_i^n + u_i^{n+1}}{2}, \quad (4.1)$$

Where $\Delta u_i^{n+1} = u_i^{n+1} - u_i^n$, $L_x(x_i, t^n) = \frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x}$ and

$$L_{xx}(x_i, t^n) = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}.$$

When we use the implicit scheme in (4.1), the appearance of nonlinear terms creates a problem. Due to nonlinearity term in (4.1), the generated system will be system of nonlinear equations. To get the approximate solution, we need to apply Newton's method. Our aim is to make the system (4.1) is linear and the resulting system will be system of linear tridiagonal equations. However, this problem can be overcome by using a Taylor series expansion, the descriptions are given in (4.2) - (4.4):

$$F_i^{n+1} = F_i^n + \Delta t \left(\frac{\partial F}{\partial t} \right)_i^n + \Delta t^2 \left(\frac{\partial^2 F}{\partial t^2} \right)_i^n + \ldots\ldots\ldots$$

$$F_i^{n+1} = F_i^n + A \Delta u_i^{n+1} + O(\Delta t^2), \quad (4.2)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -L_x \frac{(F_i^n + F_i^n)}{2} + \mu L_{xx} \frac{u_i^n + u_i^{n+1}}{2},$$

$$u_i^{n+1} + \frac{1}{2} \Delta t \left[L_x (u_i^n u_i^{n+1}) - \mu L_{xx} (u_i^{n+1}) \right] = u_i^n + \frac{1}{2} \mu L_{xx} u_i^n, \quad (4.3)$$

$$L_x(u_i^n u_i^{n+1}) = \frac{u_{i+1}^n u_{i+1}^{n+1} - u_{i-1}^n u_{i-1}^{n+1}}{2\Delta x}$$ And

$$L_{xx}(u_i^{n+1}) = \frac{u_i^{n+1} - 2u_i^{n+1} + u_i^{n+1}}{\Delta x^2}.$$

Rearrange the equation (4.3) and written in tridiagonal form, we arrive at

$$a_i^n u_{i-1}^{n+1} + b_i^n u_i^{n+1} + c_i^n u_{i+1}^{n+1} = d_i^n, \qquad (4.4)$$

Where $a_i^n = -\frac{\Delta t}{4\Delta x} u_i^n - \frac{r}{2}$, $b_i^n = 1 + r$,

$c_i^n = \frac{\Delta t}{4\Delta x} u_{i+1}^n - \frac{r}{2}$, $d_i^n = \frac{1}{2} r u_{i-1}^n + (1-r) u_i^n + \frac{r}{2} u u_i^n$

and $r = \mu \frac{\Delta t}{\Delta x^2}$. The truncation error is $O(\Delta t^2, \Delta x^2)$.

A. *Numerical Experiments*

Nonlinear hyperbolic equation (2.5) is solved by Crank Nicolson implicit scheme with multigrid method with initial condition $u(x, 0) = \sin(2\pi x)$, $t > 0$ and the boundary condition $u(x_i, 0) = u(x_f, 0) = 0$. For unsteady flow the shock formation is provided by the sinusoidal wave that's reason we have chosen the initial condition as a sine function.

B. *Numerical Result and Discussion*

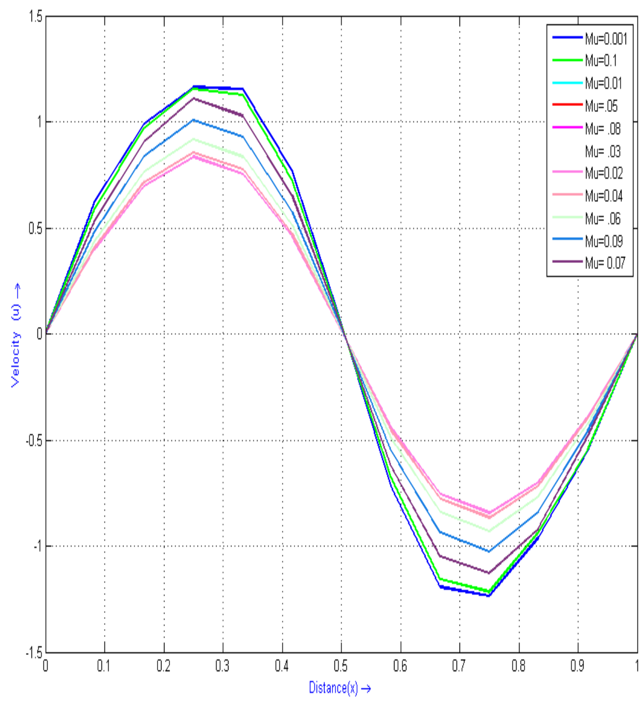

Figure 2. Numerical solution of Viscid Burgers' equation with different viscosity

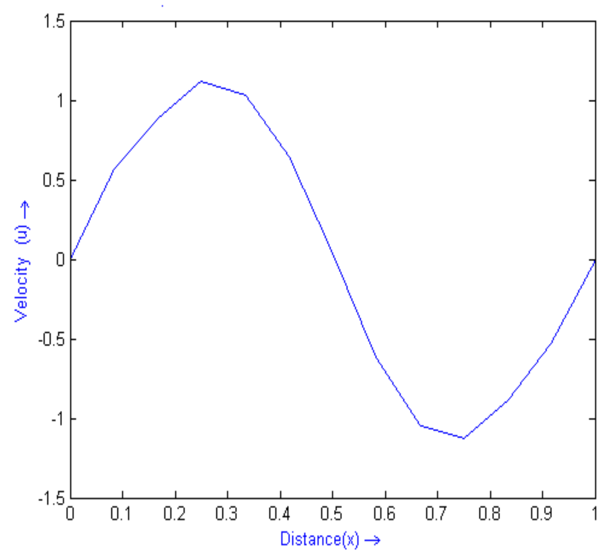

Figure 3. Numerical solution of viscid Burgers' equation with multigrid method

C. *Exact solution*

In this paper, we have implemented equation (2.5) with exact solution $u(x, t) = e^t \sin(2\pi x)$. Here viscosity μ is given by

$$\mu = -\frac{1 + 2\pi e^t \cos(2\pi x)}{4\pi^2}. \qquad (4.1)$$

In the implementation procedure, we have followed backward Euler's method for time discritization and central discritization scheme for space discritization.

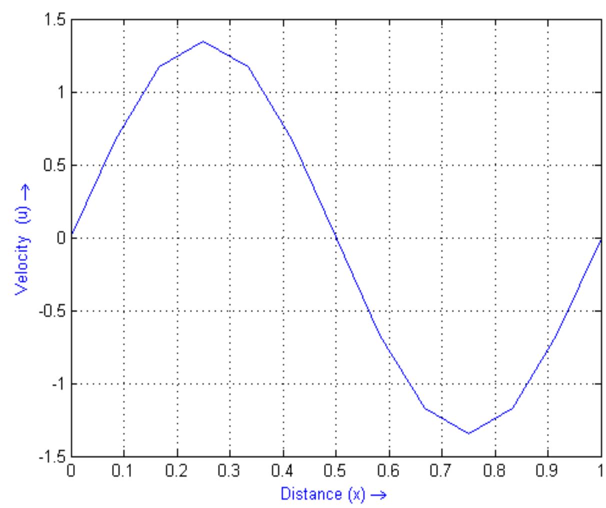

Figure 4. Profile of exact solution

In Figure 2, we have plotted the graph velocity vs. distance with different increasing viscosity, expected shock fade. We got the better result when we solve implicit scheme with multigrid method. Error is given in Table 1 and the error is removed by using of multigrid method. In this paper we use only V-cycle of multigrid method for improved result.

Table 1 when time step $\Delta t = 0.01$

Table I

Grid	Error = $\|\| u_E-u_h \|\|$	Error = $\|\|u_E-u_{h+M}\|\|$
51X51	1.5114	1.2026
101X101	7.6652	7.7848
301X301	38.6014	38.6118
601X 601	43.7403	43.7397

Table 2 when time step $\Delta t = 0.02$

Table II

Grid	Error = $\|\| u_E-u_h \|\|$	Error = $\|\|u_E-u_{h+M}\|\|$
51X51	1.23676	1.2675
101X101	10.3493	17.41844
301X301	29.88544	29.88394
351X 351	40.7719	40.77144

In Table 1 and 2, we have considered u_E is the exact solution, u_h is the approximate solution based on implicit method and Crank Nicolson scheme and u_{h+M} is the approximate solution with multigrid method. Column I error is always greater than column II for respective grid size, its means that our results are improved when I use multigrid method.

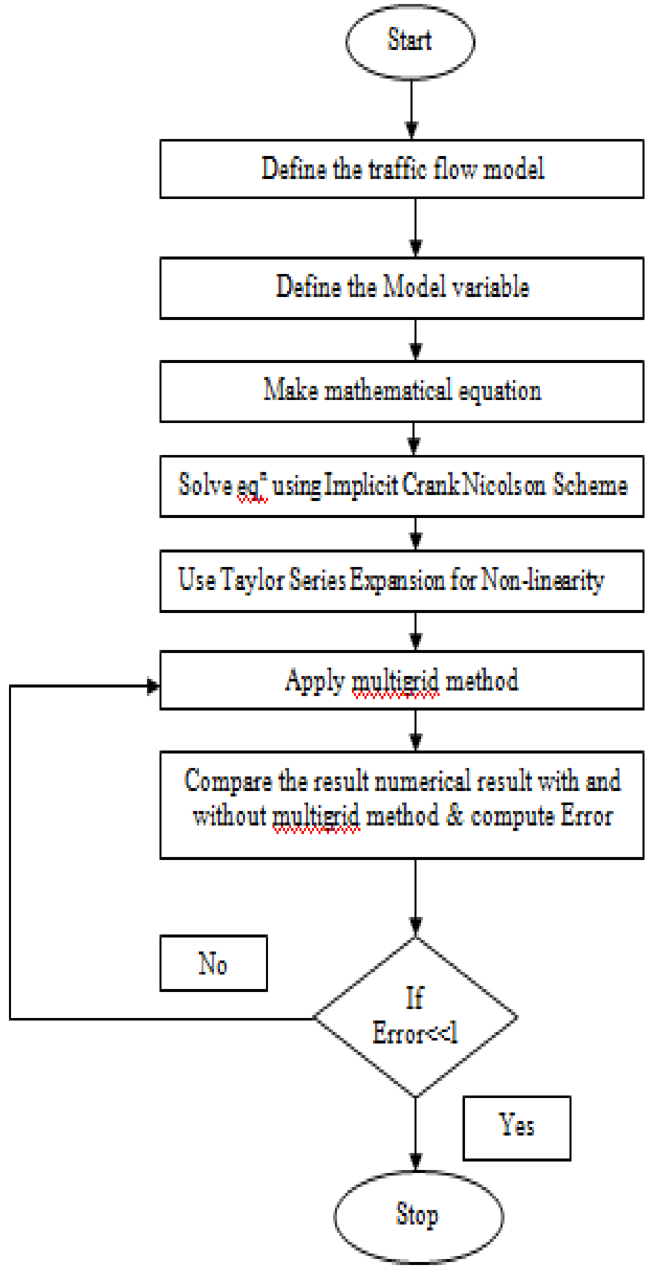

Figure 5. Flow chart of Burgers' equation solution with multigrid method

Conclusion

Implicit finite difference scheme for one dimensional viscous Burgers' equation has been presented using Taylor series expansion. Numerical results are given in Table I and II, We got the improved results when we solve equation (2.5) using Implicit scheme with multigrid method and compare with exact solution of Burgers' equation. The advantage of this suggested method is that it is second order of accuracy with respect to time and distance. The effect of viscous term in Burgers' equation is to diminish the amplitude of shock wave with respect to time and avoid the shock formation.

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