UACEE International Journal of Advancements in Electronics and Electrical Engineering – IJAEEE
Volume 2 : Issue 2 [ISSN 2319 – 7498]
Publication Date : 05 June 2013

Solution of Periodic Burgers' Equation using Modified Implicit Scheme with Multigrid Method

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***Abstract*— in this article we discussed numerical solution of periodic Burgers' equation using the implicit scheme with multigrid method and the solutions obtained here are compared with analytical solution. We constructed the exact solution using Cole- Hopf transformation and separation of variables.**

***Keywords*- Multigrid method, *V* cycle, Burgers' equation, Implicit scheme, Cole- Hopf Transformation, Sepration of Variable.**

I. *Introduction*

The research on burgers' equation was started seven decades back by J. M. Burgers and later Cole and hopf extended his work. In this paper, we discussed the solution of viscid burgers' equation using implicit scheme. Here we modify the implicit scheme by using the average of jacobian matrix value at *n* and *n+1* time level in the burger equation. In our analysis the backward difference scheme has been used for discretization of time direction and central difference scheme for discretization of space direction. The multigrid method solves the algebraic linear system by iteration [1-4]. Multigrid method has three steps namely relaxation, restriction and interpolation. We have implemented the *V*-cycle algorithm for multigrid method. For exact solution we have implemented the Cole-Hopf transformation [5-8] and separation of variable [9]. Burgers' equation is easily considered as a one dimensional Navier-Stokes due to the nonlinear convection and viscosity terms.

II. *Preliminary*

The quasi linear, unsteady burgers' equation is given by the following PDEs:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=\mu u_{xx} \tag{1.1}$$

Where *f (u)* is called flux function with $f(u)=\frac{u^2}{2}$ and *μ* is a viscous term related to the Reynolds no. *R* (= 1/μ). Initial condition is *u(x, 0) = sin(x), t>0* and periodic boundary condition *u (0, t) = u (2π, t) = 0*. Equation (*1.1*) is known as a viscid Burgers' equation. When we drop the viscous term form equation *(1.1)* then it will become inviscid burgers' equation.

III. *Multigrid method*

Multigrid method [2] is an iterative method. For large problem we require the optimal solution and for optimal solution we require hierarchical algorithms like multigrid method. It is also applicable for higher dimensions.

Algorithm:

Two grid *V*-cycle:

- Iterate on $A_h.u = b_h$ to uh (say 3 Jacobi or Gauss-Seidel Step).
- Restrict the residual $u_h = b_h - A_h.u_h$ to the coarse grid by $r_{2h} = R_h^{2h}.r_h$.
- Solve $A_{2h}E_{2h} = r_{2h}$ (or come close to E_{2h} by 3 iteration from $E = 0$.
- Interpolation E_{2h} back to $E\_h = I_{2h}^{h}.E_{2h}$. Add E_h to u_h.
- Iterate 3 more times on $A_h.u = b_h$ starting from the improved $u_h + E_h$.

$A = A_h$ = original matrix.
$R = R_h^{2h}$ = restriction matrix.
$I = I^h_{2h}$ = interpolation matrix.

- Step 3 involves a fourth matrix A_{2h} to be defined now. A_{2h} is square and it is smaller than the original A_h.

The coarse grid matrix is $A_{2h} = R_h^{2h}.A_h.I^h_{2h}$

IV. *Discretization of Burgers' equation*

In this section, we have discussed the fully discrete scheme for burgers' equation. Applying implicit schemes to nonlinear equations is not as straight forward as for linear equations.

Formulation of Implicit Scheme:

Jacobian Matrix $A = \dfrac{\partial f(u)}{\partial u} = u$.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}.$$

We are replacing Jacobian matrix A by average value of n and n+1 time step

$$A = u_i^n = \frac{u_i^n + u_i^{n+1}}{2}.$$

So we get $$\frac{\partial u}{\partial t} + \frac{(u_i^n + u_i^{n+1})}{2}\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}. \qquad (3.1)$$

Using backward Euler difference scheme in time direction and central difference in space direction in equation *(3.1)*, we arrive at

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_i^n + u_i^{n+1})}{2}\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} = \mu\frac{(u_{i+1}^{n+1} - 2u_i^{n+1} - u_{i-1}^{n+1})}{\Delta x^2}. \qquad (3.2)$$

we made arrangement in equation *(3.2)*, and we obtain

$$-\mu\frac{\Delta t}{\Delta x^2}u_{i-1}^{n+1} + [1 + \frac{\Delta t}{4\Delta x}(u_{i+1}^n - u_i^n) - \mu\frac{\Delta t}{4\Delta x}]u_i^{n+1} + -\mu\frac{\Delta t}{\Delta x^2}u_{i+1}^{n+1}$$
$$= u_i^n - \frac{\Delta t}{4\Delta x}u_i^n(u_{i+1}^n - u_{i-1}^n). \qquad (3.3)$$

Rewriting the equation *(3.3)*, we get

$$a_i^n u_{i-1}^{n+1} + b_i^n u_i^{n+1} + c_i^n u_{i+1}^{n+1} = d_i^n. \qquad (3.4)$$

Where $a_i^n = -\dfrac{\Delta t}{\Delta x^2}$.

$$b_i^n = 1 + \frac{\Delta t}{4\Delta x}(u_{i+1}^n - u_{i-1}^n) - 2\mu\frac{\Delta t}{\Delta x^2}.$$

$$c_i^n = -\frac{\Delta t}{\Delta x^2}.$$

$$d_i^n = u_i^n - \frac{\Delta t}{4\Delta x}u_i^n(u_{i+1}^n - u_{i-1}^n),.$$

And the truncation error is $O(\Delta t^2, \Delta x^2)$.

A. *Exact solution*

Here we have used Cole-Hopf Transformation for reducing the Burger equation into heat equation. The Cole [5]-Hopf [6] Transformation is defined by

$$u = -\mu\frac{\phi_x}{\phi}. \qquad (4.1)$$

From equation (4.1)

$$u_t = 2\mu \frac{(\phi_t \phi_x - \phi\phi_{xt})}{\phi^2}.$$

$$uu_x = 4\mu^2 \frac{\phi_x(\phi\phi_{xx} - \phi_x^2)}{\phi^3}.$$

$$\mu u_{xx} = \frac{-2\mu^2(2\phi_x^3 - 3\phi\phi_{xx}\phi_x + \phi^2\phi_{xxx})}{\phi^3}.$$

Now we will substitute the value of u_t, uu_x, μu_{xx} in equation *(1.1)*, we get

$$-\phi\phi_{xt} + \phi_x(\phi_t - \mu\phi_{xx}) + \mu\phi\phi_{xxx} = 0.$$

$$\phi_x(\phi_t - \mu\phi_{xx}) = \phi(\phi_t - \mu\phi_{xx})_x.$$

$$(\phi_t - \mu\phi_{xx}) = 0. \qquad (4.2)$$

This equation is called heat equation.

$$u = -2\mu(\log\phi)_x.$$

Hence, $\phi(x,t) = e^{-\int \frac{u(x,t)}{2\mu}dx}$. (4.3)

Using the initial condition $\phi(x,0) = e^{-\int \frac{u(x,0)}{2\mu}dx}$ and the Boundary condition $\phi(x_i,t) = \phi(x_f,t) = 0$, now we can solve the heat equation using separation of variable [9].

Here initial condition is a function of *x*, we are assuming *Φ(x, t)* is separable in x and time *t* and defined as

Φ(x, t) = X (t).T(t).

Now we are putting respective derivative in equation *(4.2)*

$$X.T'(t) = X''(x).T(t).$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k.$$

Here, *K* is separation constant.

$T'(t) = K.T(t)$ And solution of this equation is obtained in the form of

$T(t) = T_0 e^{kt}$, $k = -m^2$.

We are taking *k* as negative because *T (t)* will increase if *k* is positive and decrease only when *k* is negative. Now from equation *(4.2)*, we obtain

$$X'' + m^2 X = 0. \qquad (4.4)$$

Equation *(4.4)* is a simple harmonic equation so its solution will be

$X(x) = A \cos(m x) + B \sin(m x)$. (4.5)

Now we apply the boundary condition

$X(x_i, 0) = 0$, $X(x_f, 0) = 0$ in equation *(4.5)*, we arrive at

$\sin(mx_f) = 0$, $m.x_f = n\pi$.

The general exact solution of equation *(1.1)* is

$$\phi(x,t) = \sum_{n=0}^{\infty} B_n e^{-\frac{n^2\pi^2 t}{x_f}} \sin\frac{n\pi x}{x_f}. \qquad (4.6)$$

Where $B_n = \frac{2}{x_f}\int_0^{x_f} \phi(x,0)\sin\frac{n\pi x}{x_f}dx.$

V. *Numerical Result*

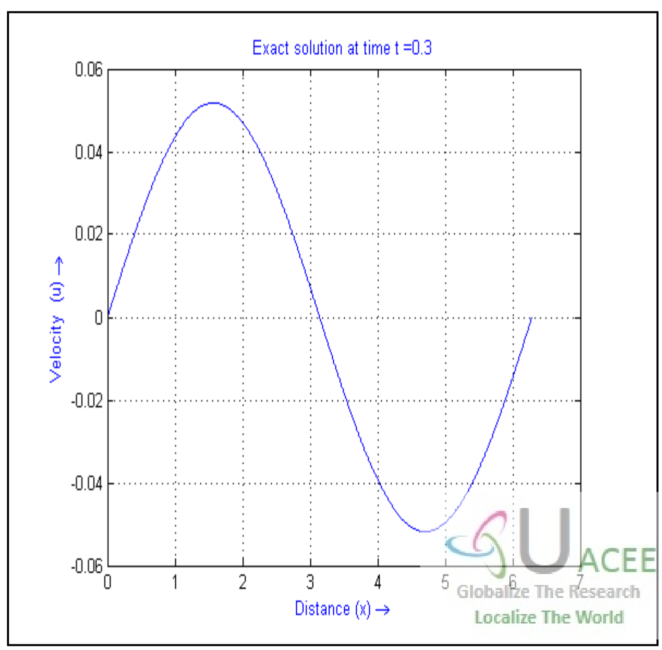

Figure 1. Exact solution

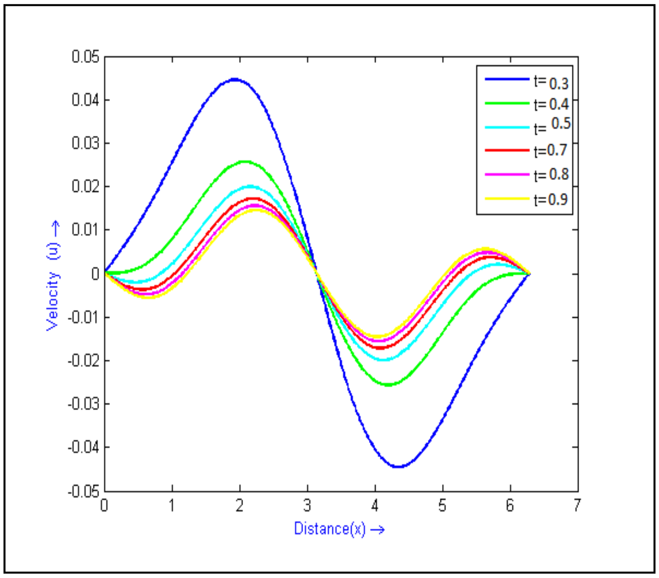

Figure 2. Solution of Non-linear Burgers' equation using modified implicit scheme

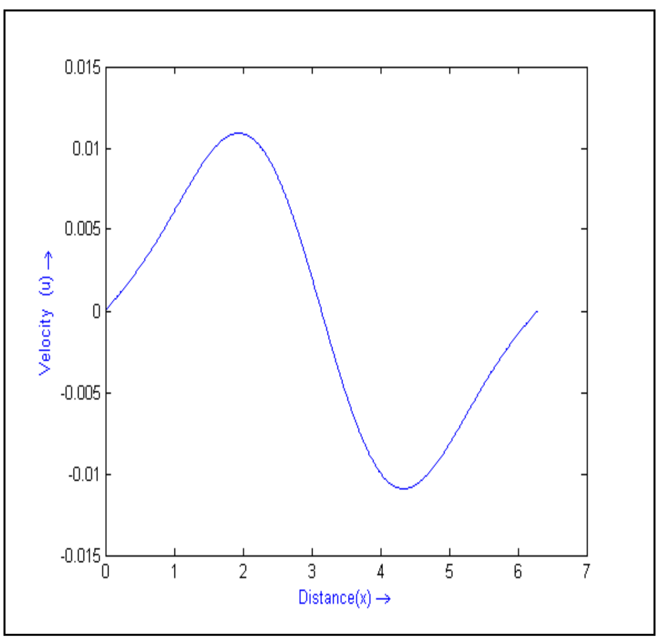

Figure 3. Improved result with multigrid method

In figure *2*, we plot the graph distance vs. velocity with different increasing time, expected shock will appear at top and bottom position.

Conclusions

In this article, we have studied the continuous numerical solution with periodic initial condition. It is well known that burgers' equation is reducible into heat equation and after that we have many analytical methods to solve the heat equation. This fact had motivated us to write the exact solution. We observe that by using modified implicit scheme we get results similar to exact solution. Then by applying multigrid method we are able to improve our result.

Acknowledgment

The author (Murari Sharan) would like to thank Professor Dr Samir k Das, Lt Cdr Venkada Prasad, Nilam chirame, J V Ramana Reddy, and Naresh Kumar Nedunuri for their useful discussion. This work is supported by the DRDO at Defence Institute of Advanced Technology, India.

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