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Interdependent Queueing Model with Fixed Size Service

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Abstract
In this paper we extend the single server interdependent queuing models to bulk service queueing models. Here we consider the costumers/units served a batch of k at a time and if not, the server waits until such time to start. In this model various system characteristics like average queue length, variability of the system size are obtained.

Keywords: **Queuing system, service process, arrival process, bulk service, interdependence, joint probability, marginal probabilities.**

1. INTRODUCTION

The single server models discussed so far with the interdependent structure assumed that service mechanism is single. Here we consider a different problem which is a generalization of the earlier queueing models. Here we consider the costumers served a batch of k at a time and if not, the server waits until such time to start. A typical situation of this model is transportation of men and material, where the transhipment is the service, men and material are the customers. For this sort of situations in order to have optimal transshipment, it is appropriate to approximate the situation with the model having the interdependent arrival and service processes. For developing these interdependent models with bulk service rule, we make use of the dependence structure given by Rao K .S (1986).

2. M / M^(k)/ 1 INTERDEPENDENT QUEUEING MODEL WITH FIXED BATCH SIZE

In this section, we consider the single server queueing system having the interdependent arrival and processes with bulk service. Here we consider that the costumers are served a batch of K at a time and if not, the server waits until such times to start.
In this sort of systems the interdependence can be induced by considering the dependence structures having a bivariate Poisson distribution of the form

$$P\left[X_1 = x_1, X_2 = x_2 / t\right] = e^{-(\lambda+\mu-\in)t} \sum_{j=0}^{\min.(x_1,x_2)} \frac{(\in t)^j \left[(\lambda-\in)t\right]^{x_1-j} \left[(\mu-\in)t\right]^{x_2-j}}{j!\ (x_1-j)!\ (x_2-j)!}$$

$$x_1, x_2 = 0, 1, 2, \ldots \quad \text{and} \quad 0 < \lambda, \mu, \in < \min(\lambda, \mu) \qquad \ldots (1)$$

$P[X_1=x_1, X_2=x_2 / t]$ is the joint probability of x_1 arrivals and x_2 services during time t.
The marginal distribution of arrival and services are Poisson with parameters λ and μ respectively .Thus inter arrival times and service times follow negative exponential distributions of the form $\lambda e^{-\lambda t}$ and $\mu e^{-\mu t}$ respectively where λ is the mean arrival rate and μ is the mean service rate (Feller 1969).
∈ is the covariance between the number of arrivals and services at time t. This dependence structure turns out to be independent structure if ∈ = 0 (Teicher1954).

3. POSTULATES OF THE MODEL

The postulates of the model with this dependence structure are
1. The occurrence of the events in non-overlapping time intervals are statistically independent.
2. The probability that no arrivals and no service completions occur in an infinitesimal interval of time Δt is
$1 - [(\lambda + \mu - \in) t] + O(\Delta t)$
3. The probability that no arrival and one service completion occurs in Δt is
$(\mu - \in) t + O(\Delta t)$
4. The probability that one arrival and no service completion occurs in Δt is
$(\lambda - \in) t + O(\Delta t)$
5. The probability that one arrival and no service completion occurs in Δt is $\in t + O(\Delta t)$.
This postulate is due to the dependence structure between the arrivals and service completions.
6. The probability that the occurrence of an event other than the above events during Δt is O(Δt)

There is equivalence between the postulates and the process. Further for given values of λ, μ the covariance $\in = r\sqrt{\lambda \mu}$, where *r* is the correlation coefficient between arrivals and services. Since ∈ is a function of *r*, throughout this paper ∈ treated as dependence parameter. This is the structure given by Rao K.S (1986).
Let $P_n(t)$ be the probability that there are n customers/units in the system at time t.

The difference differential equation of the model are

$$P_n^{/}(t) = (\mu - \in) P_{n+k}^{/}(t) - (\lambda - \mu + 2\in) P_n^{/}(t) + (\lambda - \in) P_{n-1}^{/}(t) \quad \text{for } n \geq k$$

$$P_n^{/}(t) = (\mu - \in) P_{n+k}^{/}(t) - (\lambda - \in) P_n^{/}(t) + (\lambda - \in) P_{n-1}^{/}(t) \quad \text{for } 1 \leq n \prec k \qquad \ldots (2)$$

$$P_0^{/}(t) = -(\lambda - \in) P_0^{/}(t) + (\mu - \in) P_k^{/}(t) \quad \text{for } n=0 \quad \ldots(3)$$

Assuming that the system is reached study state, the transition equations of model are

$$(\mu - \in) P_{n+k} - (\lambda + \mu - 2\in) P_n + (\lambda - \in) P_{n-1} = 0 \quad \text{for } n \geq k$$

$$(\mu - \in) P_{n+k} - (\lambda - \in) P_n + (\lambda - \in) P_{n-1} = 0 \quad \text{for } 1 \leq n \prec k \qquad \ldots (4)$$

$$-(\lambda - \in) P_0 + (\mu - \in) P_k = 0 \quad \text{for } n=0 \quad \ldots (5)$$

Using the non homogeneous linear difference equation technique, we obtain

$$p_n = \begin{cases} P_0 \dfrac{1-r^{n+1}}{1-r} & 1 \le n \prec k \\ P_0\left(\dfrac{\lambda-\in}{\mu-\in}\right) r^{n-k} & n \ge k \end{cases} \quad \ldots (6)$$

Where r is the root which lie in the interval (0,1) of the characteristic equation

$$\{(\mu-\in)D^{k+1} - (\lambda+\mu-2\in)D + (\lambda-\in)\}P_n = 0 \qquad \text{for } n \ge k \quad \ldots (7)$$

Where D is the operator.

Using the bounded conditions $P_i \ge 0$ and $\sum P_n = 1$

We obtain that $P_0 = \dfrac{1-r}{k}$ … (8)

The Probability that there are n customers in the system at any arbitrary time is

$$p_n = \begin{cases} \dfrac{1}{k}(1-r^{n+1}) & 1 \le n \prec k \\ \left(\dfrac{1-r}{k}\right)\left(\dfrac{\lambda-\in}{\mu-\in}\right) r^{n-k} & n \ge k \end{cases} \quad \ldots (9)$$

Where *n* is as given in the equation (6)

Now the probability generating function of the model is

$$P(Z) = \frac{(1-z^k)\sum p_n z^n}{\rho z^{k+1} - (p+1)z^k + 1} \qquad \text{Where } \rho = \frac{\lambda-\in}{\mu-\in}$$

Using Rouche's theorem if we denote the root of the characteristic equation which lies out side the unit circle is Z_0 then

$$P(Z) = \frac{(z_0-1)\sum z^n}{(z_0-z)k}$$

$$\text{Where } z_0 = \frac{1}{r} \quad \ldots (10)$$

From equation (9), we observe that there is recursive relation for the probability mass function

$$p_n = p_k r^{n-k} \quad n \ge k \quad \text{where r is given above equation (6)}$$

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4. MEASURES AND EFFECTIVENESS

The probability of the system is empty if $P_0 = \frac{1-r}{k}$

Where r is given in the equation (6)

For various values of ∈ and k for given value of λ and μ we computed P_0 values and are given in the table (1.2) the values of P_0 for fixed k, ∈ for varying λ, μ are given in the table (1.4)

Table 1.1 The values of r : where λ = 2 and μ = 4

k \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.4872	0.4737	0.4595	0.4444	0.4286	0.4118	0.3939	0.375	0.3548
2	0.366	0.3586	0.3507	0.3423	0.3333	0.3238	0.3135	0.3025	0.2906	0.2777
3	0.3425	0.3362	0.3294	0.3222	0.3145	0.3061	0.2972	0.2875	0.277	0.2656
4	0.3362	0.3302	0.3238	0.317	0.3097	0.3018	0.2932	0.2839	0.2738	0.2628
5	0.3343	0.3284	0.3222	0.3155	0.3083	0.3005	0.2921	0.283	0.273	0.2621
6	0.3336	0.3279	0.3217	0.315	0.3079	0.3002	0.2918	0.2827	0.2728	0.262

Table 1.2 The values of P_0 λ = 2 and μ = 4

K \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.5128	0.5263	0.5405	0.5556	0.5714	0.5882	0.6061	0.625	0.6452
2	0.317	0.3207	0.32465	0.32885	0.33335	0.3381	0.34325	0.34875	0.3547	0.36115
3	0.219167	0.221267	0.223533	0.225933	0.2285	0.2313	0.234267	0.2375	0.241	0.2448
4	0.16595	0.16745	0.16905	0.17075	0.172575	0.17455	0.1767	0.179025	0.18155	0.1843
5	0.13314	0.13432	0.13556	0.1369	0.13834	0.1399	0.14158	0.1434	0.1454	0.14758
6	0.111067	0.112017	0.11305	0.114167	0.11535	0.116633	0.118033	0.11955	0.1212	0.123

Table 1.3: The values of r: k = 2 and ∈ = 0.2

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.0759	0.1585	0.2319	0.2986	0.3601	0.4175	0.4715	0.5227	0.5714
11	0.0693	0.1455	0.2136	0.2758	0.3333	0.3872	0.4379	0.486	0.5319
12	0.0637	0.1345	0.1981	0.2563	0.3104	0.3611	0.409	0.4545	0.4979
13	0.059	0.125	0.1847	0.2395	0.2906	0.3385	0.3839	0.427	0.4682
14	0.055	0.1168	0.173	0.2248	0.2732	0.3187	0.3618	0.4029	0.4422
15	0.0514	0.1096	0.1627	0.2119	0.2578	0.3012	0.3423	0.3815	0.419

Table 1.4: The values of P_0 k = 2 and $\in = 0.2$

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.46205	0.42075	0.38405	0.3507	0.31995	0.29125	0.26425	0.23865	0.2143
11	0.46535	0.42725	0.3932	0.3621	0.33335	0.3064	0.28105	0.257	0.23405
12	0.46815	0.43275	0.40095	0.37185	0.3448	0.31945	0.2955	0.27275	0.25105
13	0.4705	0.4375	0.40765	0.38025	0.3547	0.33075	0.30805	0.2865	0.2659
14	0.4725	0.4416	0.4135	0.3876	0.3634	0.34065	0.3191	0.29855	0.2789
15	0.4743	0.4452	0.41865	0.39405	0.3711	0.3494	0.32885	0.30925	0.2905

From tables (1.2), (1.4) and equation (8) we observe that for fixed values of λ, μ and ∈ the values of P_0 decreases as k increases. As the dependence parameter ∈ increases the value of P_0 increases for fixed values of λ, μ and k.

The value of P_0 decreases for fixed values of μ, k and ∈ as λ increases. As μ increases the value of P_0 increases for fixed values of μ, k and the dependence parameter ∈. If the mean dependence rate is zero then the value of P_0 is same as in the $M / M^{(k)} / 1$ model.

The average numbers of customer in the system are obtained

$$L_s = \sum_{n=0}^{\infty} np_n = \frac{k-1}{2} + \frac{r}{1-r} \qquad \text{. . . (12)}$$

The average number of customers in the queue

$$L_q == \frac{k-1}{2} + \frac{r}{1-r} - \frac{k-1+r}{k} \qquad \text{. . . (13)}$$

Where *r* is given in the equation (6)

The values of L_s and L_q are computed and are given in the tables (1.5) and (1.7) for given values of λ, μ and for various values of ∈ and k respectively. The values of L_s and L_q are fixed values of ∈, k and for varying values of μ and λ are also given in tables (1.6) and (1.8)

Table 1.5 Values of L_s λ = 2 and μ = 4

k \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1	0.950078	0.900057	0.850139	0.799856	0.750088	0.700102	0.649893	0.6	0.549907
2	1.077287	1.059089	1.04012	1.02045	0.999925	0.978852	0.956664	0.933692	0.909642	0.884466
3	1.520913	1.506478	1.491202	1.475361	1.458789	1.44113	1.42288	1.403509	1.383126	1.361656
4	2.006478	1.992983	1.978852	1.964129	1.948646	1.932254	1.914827	1.896453	1.877031	1.856484
5	2.502178	2.488982	2.475361	2.46092	2.445713	2.429593	2.412629	2.3947	2.375516	2.355197
6	3.0006	2.987874	2.974274	2.959854	2.944878	2.92898	2.91203	2.894117	2.875138	2.855014

Table 1.6 The values of L_s k = 2 and ∈ = 0.2

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.582134	0.688354	0.801914	0.92572	1.062744	1.216738	1.392148	1.595118	1.833178
11	0.57446	0.670275	0.771617	0.880834	0.999925	1.131854	1.279043	1.445525	1.636296
12	0.568034	0.655402	0.747038	0.844628	0.950116	1.06519	1.192047	1.333181	1.491635
13	0.562699	0.642857	0.726542	0.814924	0.909642	1.011716	1.123113	1.245201	1.380406
14	0.558201	0.632246	0.70919	0.78999	0.875894	0.967782	1.066907	1.174761	1.292757
15	0.554185	0.623091	0.694315	0.768875	0.847346	0.931025	1.02045	1.116815	1.22117

Table 1.7 Values of L_q λ = 2 and μ = 4

k \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	0.5	0.462878	0.426357	0.390639	0.355456	0.321488	0.288302	0.255993	0.225	0.195107
2	0.394287	0.379789	0.36477	0.3493	0.333275	0.316952	0.299914	0.282442	0.264342	0.245616
3	0.740079	0.727745	0.714735	0.701295	0.687289	0.67243	0.657147	0.641009	0.624126	0.606456
4	1.172428	1.160433	1.147902	1.134879	1.121221	1.106804	1.091527	1.075478	1.058581	1.040784
5	1.635318	1.623302	1.610921	1.59782	1.584053	1.569493	1.554209	1.5381	1.520916	1.502777
6	2.111667	2.09989	2.087324	2.074021	2.060228	2.045613	2.030064	2.013667	1.996338	1.978014

Table 1.8 Values of L_q k = 2 and ∈ = 0.2

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.044184	0.109104	0.185964	0.27642	0.382694	0.507988	0.656398	0.833768	1.047478
11	0.03981	0.097525	0.164817	0.242934	0.333275	0.438254	0.560093	0.702525	0.870346
12	0.036184	0.088152	0.147988	0.216478	0.294916	0.38464	0.487547	0.605931	0.742685
13	0.033199	0.080357	0.134192	0.195174	0.264342	0.342466	0.431163	0.531701	0.646306
14	0.030701	0.073846	0.12269	0.17759	0.239294	0.308432	0.386007	0.473311	0.571657
15	0.028485	0.068291	0.112965	0.162925	0.218446	0.280425	0.3493	0.426065	0.51167

From equations (12), (13) and from the corresponding tables we observe that as ∈ increases the values of L_s and L_q are decreasing and also as k increases the value of L_s and L_q are increasing for fixed values of other parameters.

As the arrival rate increases the values of L_s and L_q are increasing for fixed values of μ, k and ∈. As μ increases the values of L_s and L_q are decreasing for fixed values of λ, k and ∈. When the dependence parameter ∈=0 then the average queue length is same as that of $M/M^{(k)}/1$ model. When k=1 this is same as M/M/1 interdependence model.

The variability of this model can be obtained as

$$V = \frac{k^2 - 1}{12} + \frac{r}{(1-r)^2} \qquad \ldots (14)$$

And the coefficient of variation of the model is

$$C.V = \frac{(1-r)\left[\left(k^2 - 1\right)(1-r) + 12r\right]}{3\left[(k-1)(1-r) + 2r\right]} \qquad \ldots (15)$$

The values of the variability of the system and coefficient of variation for various values of k, ∈ for fixed values of λ, μ are computed which are given in table (1.9) and (1.11). The values of the variability of the system and coefficient of variation for fixed values of k, ∈ and for various values of λ, μ are given in table (1.10) and (1.12).

Table 1.9 Values of V λ = 2 and μ = 4

k \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	1.852726	1.71016	1.572875	1.439626	1.312719	1.190245	1.072253	0.96	0.852305
2	1.160547	1.121671	1.08185	1.041318	0.99985	0.958152	0.915206	0.87178	0.827448	0.782281
3	1.458929	1.429664	1.399148	1.367997	1.335943	1.302392	1.268374	1.232995	1.196578	1.159117
4	2.012998	1.986015	1.958152	1.929544	1.899928	1.869098	1.836909	1.803628	1.769184	1.733565
5	2.754361	2.728084	2.70133	2.673368	2.644374	2.614142	2.582892	2.550488	2.516528	2.481362
6	3.667868	3.642561	3.615876	3.587986	3.559461	3.52967	3.498466	3.466112	3.432532	3.397715

Table 1.10 Values of V k = 2 and ∈ = 0.2

μ \ λ	1	2	3	4	5	6	7	8	9
10	0.33888	0.473831	0.643066	0.856957	1.129425	1.480452	1.938075	2.544403	3.360541
11	0.330004	0.449269	0.595394	0.775869	0.99985	1.281093	1.635951	2.089543	2.677463
12	0.322662	0.429551	0.558066	0.713397	0.90272	1.13463	1.420977	1.77737	2.224975
13	0.31663	0.413265	0.527864	0.664102	0.827448	1.023569	1.261383	1.550525	1.905521
14	0.311588	0.399735	0.50295	0.624084	0.767191	0.936602	1.13829	1.380064	1.671221
15	0.307121	0.388242	0.482073	0.591168	0.717995	0.866807	1.041318	1.247275	1.491257

Table 1.11 Values of C. V $\lambda = 2$ and $\mu = 4$

k \ ∈	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	1	1.0256	1.0526	1.081	1.1112	1.1428	1.1764	1.2122	1.25	1.2904
2	0.973742	0.979993	0.986473	0.99314	1.000025	1.006996	1.014201	1.021483	1.028866	1.036275
3	1.026796	1.033846	1.041397	1.049325	1.057727	1.066802	1.076315	1.086563	1.097514	1.109238
4	1.162186	1.17156	1.181543	1.192132	1.203477	1.21573	1.22904	1.2434	1.258956	1.275853
5	1.3314	1.3432	1.3556	1.369	1.3834	1.399	1.4158	1.434	1.454	1.4758
6	1.517874	1.531652	1.546646	1.562859	1.580049	1.598702	1.619065	1.64114	1.665173	1.691411

Table 1.12 Values of C.V $k = 2$ and $\in = 0.2$

μ \ λ	1	2	3	4	5	6	7	8	9
10	1.054482	1.071759	1.057283	1.02396	0.97874	0.925631	0.867185	0.804987	0.740299
11	1.051335	1.071575	1.063221	1.037312	1.000025	0.954894	0.904465	0.85021	0.793164
12	1.048441	1.070718	1.067081	1.047147	1.016297	0.9779	0.934107	0.886414	0.835895
13	1.045852	1.069444	1.069518	1.054392	1.028866	0.99608	0.957918	0.915917	0.870976
14	1.043531	1.067938	1.07094	1.059761	1.03871	1.01061	0.977311	0.940063	0.89986
15	1.041349	1.066297	1.071632	1.063698	1.046444	1.022315	0.99314	0.960096	0.924113

From equations (14) & (15) and from the corresponding tables we observe that as μ increases the variability of the system size decreases and the coefficient of variation decreases.

As λ increases and for fixed values of μ, $\in$ and k, the variability of the system size increases and the coefficient of variation increases. We also observe that as $\in$ increases the variability of the system size decreases and the coefficient of variation increases for fixed values of λ, μ and k.

As k increases, the variability of the system increases and the coefficient of variation increases.

For $\in=0$ and k=1, this model reduces to M/M/1 classical model. The mean queue length and variability of the system size of this model are increase that of the classical model. When k=1, this model becomes M/M/1 interdependent model, for $\in=0$, this model is same as $M/M^{(k)}/1$ model.

CONCLUSION

In this paper, we extend the single server interdependent queueing models to bulk service queueing models with fixed size service. Where the arrival and service patterns are interdependent. These models have wider applicability in transportation, inventory control, machine interference problems, neurophysiological systems etc. For efficient design and to predict the system performance measures, the system behavior is analyzed by using the system characteristics like average number of costumers in the system and queue, variability of the system size, probability of the system emptiness and coefficient of variation of the system.

In this paper, it is observed that the positive dependence between the arrival and the service completions can reduce the mean queue length and variability of system size. This is a more useful result in developing the optimal operating policies.

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