Closed form analytical model of Cross-Beam Resonator and its Verification.

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***Abstract*— The objective of this work is to develop analytical solutions for analyzing a static behavior of a new micro-electro-mechanical (MEMS) resonator called Cross-Beam Resonator in terms of its geometry, material properties and the excitation voltage. Analytical equations have been developed for the determination of natural resonant frequency, electrostatic deflection of the beam and the shift in resonant frequency due to the bias voltage. Analytical equations are verified for their accuracy by simulating the resonator using Intellisuite. It is observed that the results of analytical solution compare well with those of Intellisuite simulation. However the correction factor has been determined by curve fitting to eliminate errors. In this work the concept of clamped-clamped beam resonator has been extended to cross-beam resonator. For the purpose of analysis cross beam is considered as a grid consisting of 4 members called arm. Using the theory of matrix analysis of framed structures we analyze the structure to find the mechanical stiffness and the deflection at the center.**

***Keywords*— Cross-Beam Resonator, MEMS, Resonant Frequency, Analytical Model**

I. INTRODUCTION

Micro-electro-mechanical systems (MEMS) devices due to their characteristics can create the miniaturized, low power and inexpensive integrated RF filters. Micromechanical resonators can be used as a filtering element in electronics because of their vibrational transfer function. A vibrating beam will deflect maximum at its natural frequency than at any other frequency, giving rise to a resonant peak. This property in the transfer function of a micro-resonator leads to its use as a filter. Electro-statically actuated resonant micro-electro-mechanical systems (MEMS) devices have gained significant importance due to their geometric simplicity, broad applicability and well understood phenomenon. Several works [1], [2], [3], [4] have demonstrated filters and oscillators using capacitively transduced poly-silicon surface micro-machined vibrating resonators of various structures namely clamped-clamped beam, wine-glass disk, comb-driven folded flexure, crab-leg structure etc. Clamped-clamped beam resonator has been demonstrated previously [1], [2] achieving resonant frequency in low-frequency (LF) range. Using clamped-clamped beam a high-frequency (HF) resonator can be achieved by reducing the length but at the cost of voltage sensitivity which in turn decreases tune-ability. In this work the concept of clamped-clamped beam has been extended to cross-beam which achieves the resonant frequency in HF range at the same time maintaining good sensitivity to voltage resulting in good tune-ability. For the purpose of analysis cross beam resonator structure is considered as a grid consisting of 4 members called arm (all have same dimensions and material properties) that are rigidly joined at point E as shown in figure 1.

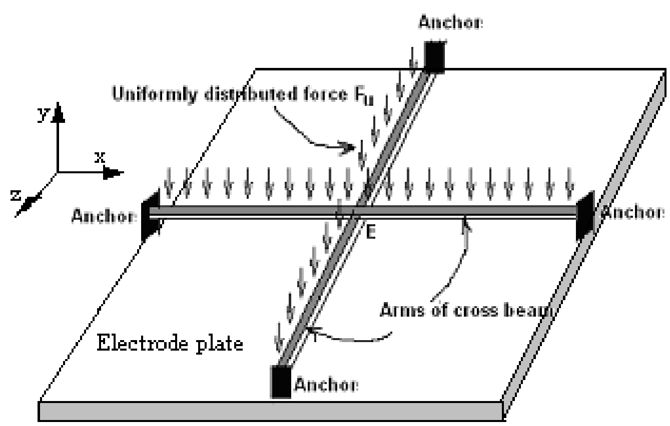

Figure 1. Cross-beam viewed as a grid

II. NATURAL RESONANT FREQUENCY OF CROSS-BEAM

Natural resonant frequency is governed by expression [5]

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{K_m}{M_{eff}}} = 0.5032\sqrt{\frac{E}{\rho(4L+w)}}\frac{h}{(L+w/2)^{3/2}} \qquad (1)$$

Effective mass M_{eff} is calculated for cross beam from kinetics and mechanical Stiffness (spring constant) of the cross beam K_m is obtained from the stiffness matrix of the grid. For cross-beam structure, M_{eff} = 0.4063m where M is the actual mass of the beam.

III. STATIC ANALYSIS

A. Displacement at the centre

Displacement at the centre of the beam in vertical direction is expressed as

$$y(L_{eff}) = S^{-1}(A_D - A_{DL}) \tag{2}$$

S is a stiffness Matrix, vector A_D represents actions in the beam corresponding to the unknown displacement which is a null vector as there are no concentrated forces or couples at the joint 'E', vector A_{DL} represents restraint actions due to force F_u in vertical direction expressed as

$$A_{DL} = 2F_u L_{eff}\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{3}$$

Substituting stiffness matrix, vector A_D and vector A_{DL} in (3) gives an expression for displacement at the centre $y(L_{eff})$ as

$$Y(L_{eff}) = \frac{-F_u L_{eff}^4}{24EI} \tag{4}$$

F_u is the force per unit length expressed as

$$F_u = \frac{V_{dc}^2 \varepsilon_0 w}{2.d_0^2} \tag{5}$$

B. Displacement at any point on the beam

Equation for the displacement of the beam element at a distance 'x' from fixed end is derived in this section. The cross beam shown in figure 1 is modeled as four fixed-guided beams. Figure 2(b) shows one fixed-guided beam with uniformly distributed lateral force, F_u, applied to the surface. For fixed-guided beams the displacement of the beam element y(x) at a distance 'x' from fixed end of the beam is expressed as [6]

$$y(x) = \frac{-F_u(x^4 - 4x^3 L_{eff} + 4x^2 L_{eff}^2)}{24EI} \tag{6}$$

$$F_u = \frac{V_{dc}^2 \varepsilon_0 w}{2.(d_0 - y(x))^2} \tag{7}$$

F_u is the line force per unit length of the beam. From equations (6) and (7), $y(x)$ can be expressed as

$$y(x) = \frac{-V_{dc}^2 \varepsilon_0 w(x^4 - 4x^3 L_{eff} + 4x^2 L_{eff}^2)}{48EI(d_0 - y(x))^2} \tag{8}$$

Since the desired variable $y(x)$ appears on both sides of equation (8) it must be solved in an iterative method. Alternately we obtain a closed form solution which involves solving a cubic polynomial in terms of $y(x)$ as explained. Expression (8) can be written as

$$y^3(x) - 2d_0 y^2(x) + d_0^2 y(x) - k' = 0 \tag{9}$$

where $k' = \frac{\varepsilon_0 w V_{dc}^2}{48EI}(x^4 - 4x^3 L_{eff} + 4x^2 L_{eff}^2)$

Discriminant of cubic polynomial (9) is

$$4d_0^3 k' - 27k'^2$$

k' being the function of geometric dimensions and x for every possible combination of these variables and d_0 it has been observed that

$$4d_0^3 k' - 27k'^2 < 0$$

This condition ensures that there is only one real root and two complex roots. The real root being the solution $y(x)$ it can be expressed as

$$y(x) = (-g/2 + h^{1/2})^{1/3} + (-g/2 - h^{1/2})^{1/3} - b/3 \tag{10}$$

$g = \frac{2d_0^3}{27} - k'$ $\quad h = \frac{k'^2}{4} - \frac{k' d_0^3}{27}$ $\quad b = -2d_0$ and

$$k' = \frac{\varepsilon_0 w_r V_{dc}^2}{48EI}(x^4 - 4x^3 L_{eff} + 4x^2 L_{eff}^2)$$

All the arms of the beam are symmetrical in shape and dimension, and force is uniformly distributed throughout the cross beam. Therefore the equation (10) holds good for the displacement of the elements of any arm at a distance 'x' from the fixed end up to L_{eff}.

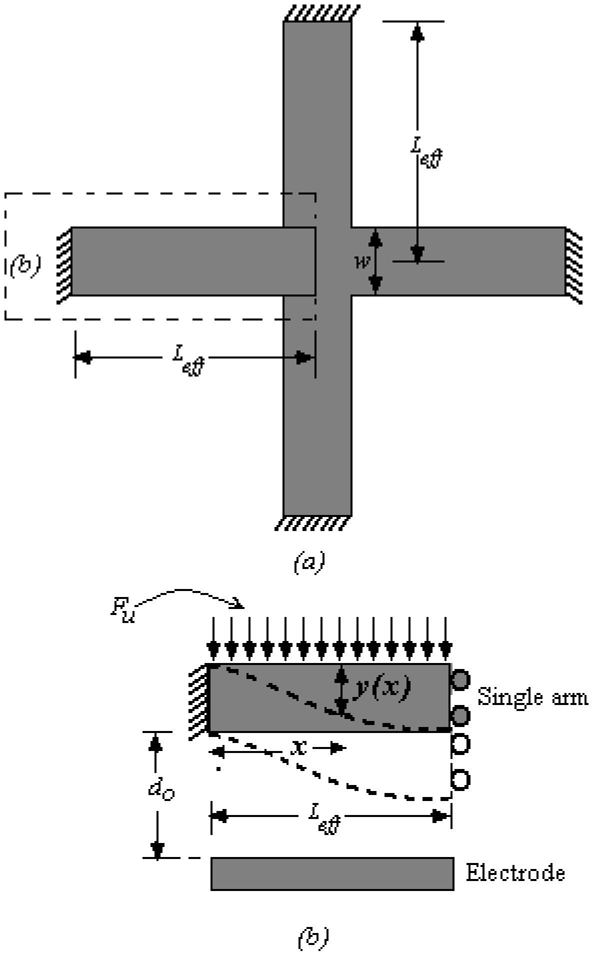

Figure 2. a) Top view of Cross-beam. b) Front view of one arm in a)

IV. SHIFT IN RESONANT FREQUENCY

There may be manufacturing variations during lithography or etching process which may vary the geometric sizes including the length, width and thickness of the beam, from design values. This in turn causes variation in resonant frequency and pass-band distortion in filters using these resonators. A well known phenomenon of electrical spring softening due to application of dc bias voltage, V_{dc}, generating electrical spring constant K_e is exploited to compensate the effects of manufacturing variations. Effectively the electrical spring constant K_e that varies with V_{dc} subtracts from the mechanical spring constant of the resonator producing an effective spring constant $K_{eff}=K_m-K_e$. This in turn lowers the resonance frequency according to the expression

$$f_r = \frac{1}{2\pi}\sqrt{\frac{K_m - K_e}{M_{eff}}} \tag{11}$$

where K_m and M_{eff} denote values at a particular location (the beam center location). Electrical spring stiffness for cantilever beam is obtained by considering the lumped model of the beam as shown in figure 3(a) where K_{eq} is the equivalent spring constant and it is the parallel combination of effective spring constant of each component (arms and the center plate) of the cross beam as shown in the figure 3(b) . Because of spring softening as explained earlier the effective spring constants can be expressed as

$K_{eff1}=K_{m1}-K_{e1}$, $K_{eff2}=K_{m2}-K_{e2}$, $K_{eff3}= K_{m3}-K_{e3}$,$K_{eff4} = K_{m4}-K_{e4}$, $K_{eff5}= K_{m5}-K_{e5}$.

Hence K_{eq} is expressed as

$$K_{eq} = K_{eff1} + K_{eff2} + K_{eff3} + K_{eff4} + K_{eff5} \tag{12}$$

K_{eff1} , K_{eff2} , K_{eff3} and K_{eff4} are the effective spring constants of the four arms and are equal since the arms are symmetrical in dimension and have the same material properties. K_{eff5} is the effective spring constant of the center plate.

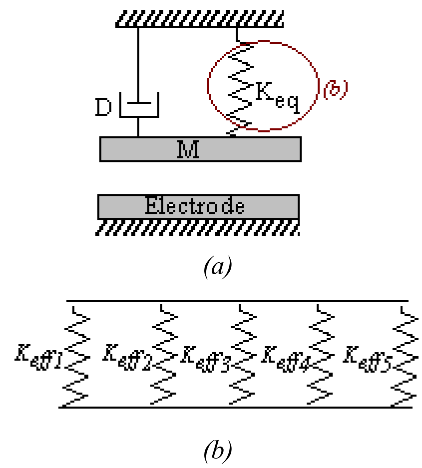

Figure 3. a)Lumped model of the Cross-beam b) Parallel combination of the effective spring constants of the arms and the center plate.

Let $K_m= K_{m1}+K_{m2}+ K_{m3}+ K_{m4}+K_{m5}$ and we know $K_{e\text{-}arm}= K_{e1}=K_{e2}=K_{e3} = K_{e4}$ equation (15) reduces to

$K_{eq} = K_m- (4\times K_{e\text{-}arm} + K_{e\text{-}plate})$. Therefore

$$K_e = 4\times K_{e\text{-}arm} + K_{e\text{-}plate} \tag{13}$$

The problem of deriving an expression for K_e is simplified to deriving equations for electric spring constants $K_{e\text{-}arm}$ and $K_{e\text{-}plate}$ of more simple structures arm and plate respectively. Consider one arm of the cross-beam shown in figure 2(b). Electrode-to-resonator gap capacitance is dependent very strongly upon the electrode-to-resonator gap spacing, which is in turn a function of the distance 'x' from the anchor. Basic expression of K_e is

$$K_e = \left|\frac{\partial F_e}{\partial y}\right|$$

where
$$F_e = \frac{1}{2}V_{dc}^2\frac{\partial C}{\partial y} \tag{14}$$

and
$$\frac{\partial C}{\partial y} = \frac{\varepsilon_0 A}{(d_0 - y(x))^2} \quad (15)$$

$y(x)$ is deflection at 'x' and d_0 is initial gap when there is no deflection. Since force is variable throughout the beam we consider an infinitesimal force at a distance 'x' from anchor $F_e(x)$. Therefore $F_e(x)$ is expressed as

$$F_e = \frac{1}{2} V_{dc}^2 \frac{\varepsilon_0 dA}{(d_0 - y(x))^2} \quad (16)$$

dA is the infinitesimal area at 'x' i.e. $dA = w \times \partial x$.

Differentiating expression (16) with respect to y gives

$$K_e = \frac{1}{2} V_{dc}^2 \varepsilon_0 w \frac{dx}{(d_0 - y(x))^3} \quad (17)$$

K_e expressed in equation (17) is electrical spring constant at a particular distance 'x' from anchor. Integrating this equation from 0 to L_{eff} gives $K_{e\text{-}arm}$.

$$K_{e-arm} = \int_0^{L_{eff}} K_e = \int_0^{L_{eff}} \frac{1}{2} V_{dc}^2 \varepsilon_0 w \frac{\partial x}{(d_0 - y(x))^3} \quad (18)$$

To obtain electrical spring constant of the center plate $K_{e\text{-}plate}$ we assume that the deflection of every point on the center plate is equal with a negligible difference. This leads to a simple expression for $K_{e\text{-}plate}$ as

$$K_{e-plate} = \frac{1}{2} V_{dc}^2 \varepsilon_0 \frac{w \times w}{(d_0 - y(L_{eff}))^3} \quad (19)$$

The total K_e is the sum of four times equation (18) and equation (19) expressed as

$$K_e = \frac{1}{2} V_{dc}^2 \varepsilon_0 w \left[4 \times \left(\int_0^{L_{eff}} \frac{\partial x}{(d_0 - y(x))^3} \right) + \frac{w}{(d_0 - y(L_{eff}))^3} \right] \quad (20)$$

V. RESULTS AND DISCUSSIONS

In this section we have compared the results of analytical solution with those of the Intellisuite simulation to verify the accuracy of analytical models. Table I presents the resonant frequency variations for three different arm lengths of the beam. Figure 4 presents the deflection at the center of the beam for different voltages obtained by analytical model and Intellsuite Simulation. During the process of deflection of the beam, the initial uniform force on the beam results in initial deflection which is different at every point on the beam.

TABLE I. RESONANT FREQUENCY FOR DIFFERENT ARM LENGTHS.

Effective Length of the arm in micrometer	Resonant Freq (Simulation results)	Resonant Freq (Analytical results)	% deviation from simulated values.
12	31.279	31.286	0.02237
14	22.824	22.838	0.0613
16	17.373	17.402	0.166

Deflection of the beam goes on increasing iteratively and force goes on building non-uniformly until equilibrium is reached. This type of deflection results in non-uniform gap $d(x) = d_0 - y(x)$ between the beam and the electrode, in turn creating a variable force as shown in the figure 5. At this state, force is maximum at the center and minimum near the anchors. In our model this effect is not taken into consideration during the derivation of analytical equations. Instead we assume a piston-

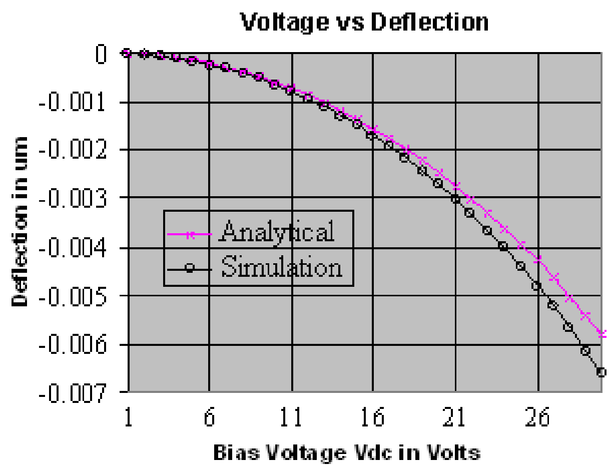

Figure 4. Vdc vs deflection

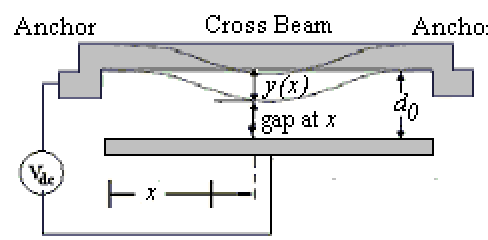

Figure 5. Cross section view of deflected cross beam

type movement of the beam towards plate resulting in uniform gap. Consequently the uniform force is assumed to be developed on the beam. This assumption results in higher value of deflection compared to those obtained from Intellisuite simulation (y axis in figure 6 is negative). Plots for the shift in the resonant frequency due to variation in the DC bias voltage obtained from Intellisuite simulation and analytical solutions are shown in figure 6 for three different lengths of beam in each case.

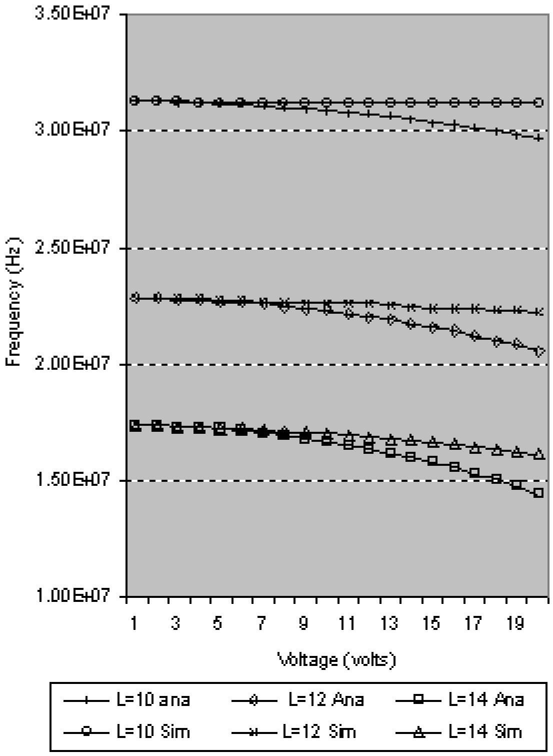

Figure 6. Resonant frequency versus Voltage plot obtained from Simulation and Analytical models.

From these plots we draw the inference that the deviation in the frequency shift increases for higher voltages, and are negligible for lower voltages. The error at a specific voltage is corrected by employing a correction factor determined by using a curve fitting tool available in MATLAB software. The polynomial equation in terms of voltage is found from curve fitting process expressed as

$$CF = P_1 \times V^2 + P_2 \times V + P_3$$

where CF is the correction factor, V is the dc bias voltage and coefficients $P_1 = -4308$, $P_2 = 9912$ and $P_3 = -1.037e + 004$.

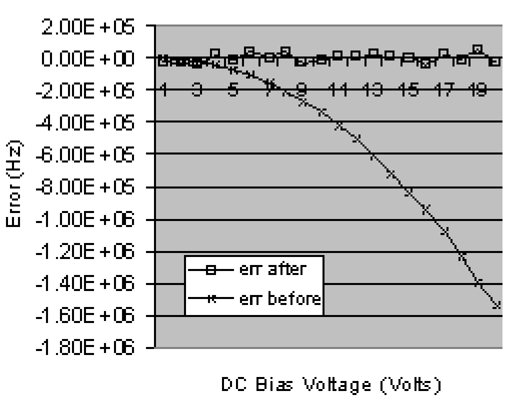

Figure 7. Comparison of Error before and after correction for L=10um

Value of CF is then deducted from the original values of the resonant frequency corresponding to the voltage at which CF is calculated. Error in the frequencies before and after applying the correction for cross-beam dimension L=10um is shown in figure. It is observed from the graph that the error reduces significantly and remains almost constant except for voltages above 17 volts in all cases.

Graph of analytically obtained corrected resonant frequency values and the resonant frequency values obtained from the simulation (required) for different voltages is plotted as shown in the figure 12 for L=10um, L=12um and L=14um.

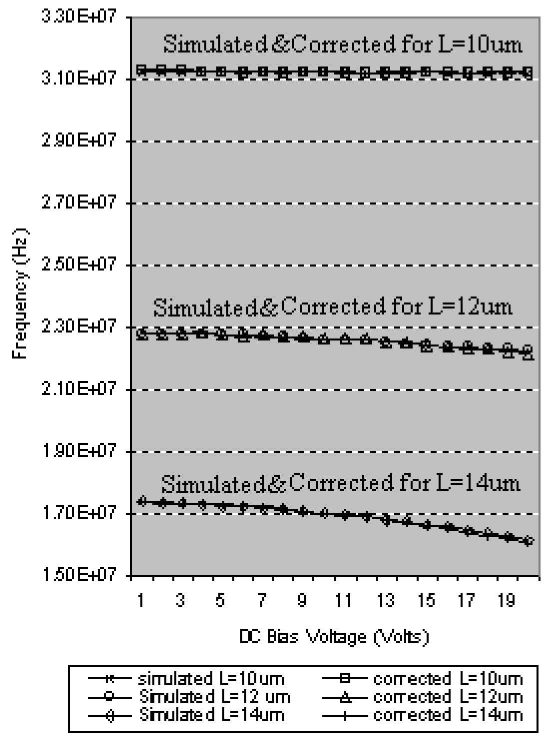

Figure 8. Analytically obtained corrected resonant frequency values and the resonant frequency values obtained from the simulation

REFERENCE

[1] Yu-Wei Lin, Seungbae Lee, Sheng-Shian Li, Yuan Xie, Zeying Ren, and Clark T-C. Nguyen (2004). "Series-Resonant VHF Micromechanical Resonator Reference Oscillators," *IEEE journal of solid-state circuits*, December 2004, vol. 39, no. 12.

[2] Frank D. Bannon, III, John R. Clark, Clark T.-C. Nguyen (2000). "High-Q HF Microelectromechanical Filters," *IEEE journal of solid-state circuits*, April 2000, vol. 35, no. 4.

[3] G.K. Fedder and Q. Jing, "A Hierarchical Circuit-Level Design Methodology for Microelectromechanical Systems," ***IEEE Trans. on Circuits & Systems 11,*** Vol. 46, No. 10, Oct 1999, pp. 1309-1315.

[4] C.T.-C. Nguyen, and R.T. Howe, "An Integrated CMOS Micromechanical Resonator High-Q Oscillator", ***IEEE J. Solid-State Circuits,*** vol. 34, No. 4, April 1999, pp.440-455

[5] Vaishali B. M, Uday Wali "Resonant Frequency of Cross-beam resonator", International Conf. on MEMS, IIT Chennai, Jan 2-5, 2009.

[6] Raymond J. Roark, Warren C. Young. *Formulas for Stress and Strain.* (Fifth Edition)